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OPTIMAL REPAIRMAN ALLOCATION MODELS

by
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ABSTRACT

A system of n components under the care of one repairman is modeled. The components are subject to failure, whereupon they may be repaired one at a time. It is desired to repair failed components in such a manner that the ergodic probability that the system works is maximized.

It is assumed that each component and the system as a whole can be either working or failed, with the relationship between the working of the system and the working of the components given by a coherent structure function. The time a component works, or the time to repair a component is an exponential random variable of known rate. All components are independent, and at most one component may be under repair at a given time.

Although the general problem is in principle soluble by known methods, computational difficulties are enormous for moderate sized systems. In addition, such methods give no general insight into the structure of the optimal policy. Therefore, bounds and approximations for general systems are highly useful.

One bound for the optimal ergodic probability that the system works is given by the ergodic probability that the system works under a particular policy. The time reversible policy given yields easily obtainable ergodic probabilities for all states, and is useful for bounding purposes.

Most real systems are highly reliable in nature. Parametrization of the rates of the exponential random variables given earlier allows investigation of asymptotic system properties as the system becomes very reliable. Specifically, for a given policy, the asymptotic ergodic probability of all states and the asymptotic passage times between states may be computed. These results allow one to obtain the asymptotic optimal unreliability of an arbitrary system, and to obtain the asymptotically optimal policy for assignment of the repairman in many cases. Intuitively, the asymptotically optimal policy is close to optimal for highly reliable systems.

Although highly unreliable systems occur less frequently, such systems may be treated in a similar manner with similar results.

Two specific examples of systems are treated in the paper: the series system and an arbitrary system of stochastically identical components. The series system occurs often in practice, since many simple systems cannot tolerate failure of any components.

These two examples lead to relatively simple solutions. Arbitrary systems generally do not. The two component parallel system yields a fairly complicated criterion for choosing between the two potentially optimal policies.

When the series system is composed of components whose failure rates are identical, the ergodic probability that the system works is independent of policy.

For the two component series system, it is optimal to repair the longer expected lifetime component first, and this is true even if the repairman is subjected to random intervals during which he is not allowed to work.

For an n component series system, the optimal policy seems to be to repair the components in order of increasing expected lifetimes. This result can be proven if the optimal policy can be written as a list, but a more general proof seems to be elusive.

When a system is composed of stochastically identical components, it is often possible to eliminate most policies from consideration. Two examples of this technique are given, including one in which the optimal policy is explicitly obtained.

TABLE OF CONTENTS

DIVISION I

Preliminaries

CHAPTER 1

Preliminaries

1.1 Introduction

The field of reliability theory has seen great activity recently. A central assumption to many models is the binary state assumption; that is, that each component and the system as a whole can assume two states: working and failed. Analysis of the relationship between the system's state and the state of the components under reasonable assumptions is the subject of coherent structure theory.[7]

Given coherent structure theory, it is easy to obtain deterministic and probabilistic models for the wear out of a system when the wear out characteristics of each component are known.[7]

An extension of the probabilistic model of the last paragraph treats a system whose components fail and are then repaired. Ross[39] treated a maintained system in which each component fails and is repaired again in accordance with an alternating renewal process. Keilson and others[12,24,27,38] deal with the same model under the more specific assumption that the time that the $i^{\underline{th}}$ component works and the time required to repair the $i^{\underline{th}}$ component are exponential random variables. Barlow[5] treats a system where other components are in a state of "suspended animation" during repair of a component. All of these maintained system models approximate a system which has separate repair resources dedicated to each component. A more realistic assumption for some systems is that the system possesses

limited repair resources which may be allocated to failed components as needed. We attempt to model such a system.

In the simplest case, the system possesses linearity of repair resources and no comparative advantage in the repair of different components by different resources. As will be shown in section 2.1, all repair resources may be optimally concentrated on a single component at any time. Thus, in this case, the system may be thought of as having a single repairman who can repair at most one component at a given time. This model will be the principal object of investigation in this paper.

Classical repairman models[2,3,4,8,10,20,22,31,33,41] deal with a system of components and spares. Upon failure, a component is replaced with a working spare, if available, and then sent into queue at the repair facility to be repaired. Usually, only one working component is treated in such a model.

For our model, if general failure and repair distributions are allowed, as in Ross,[39] things become very complicated. Such a model needs Markovian decision theory on uncountable state spaces, with all the attendant difficulties. Therefore, as in Keilson,[24] we assume that all failure and repair distributions are exponential. The memoryless property of the exponential distribution allows us to specify the system state solely in terms of the binary states of all components.

In section 1.2 we describe the assumptions and

results of coherent structure theory appropriate to our model, which is described in section 1.3. Our objective will normally be to maximize the ergodic probability that the system works. Section 1.4 proves the intuitive and important result that it is not optimal to leave the repairman idle.

General results for the model are often difficult to obtain. Division II describes three methods for approximating and boundary optimal system characteristics.

In Chapter 2 we find a particular policy for which the ergodic probabilities are easily obtainable. The optimal performance is bounded by the performance of any given policy, and in particular, is bounded by the performance of the policy given in Chapter 2. In addition, the coordinates appear to be associated under the above given ergodic probabilities, allowing a further simplification in the computation of a system bound.

Chapters 3 and 4 deal with asymptotic results for highly reliable or unreliable systems respectively. These results are obtained by assuming that the failure rates or repair rates of all components are multiplied by a common factor k, and looking at small k. The ergodic probabilities of states under arbitrary policies are asymptotically proportional to integral powers of k, allowing computation of the asymptotic optimal unreliability or reliability of the system in terms of specific constants multiplied by powers of k. An alternating renewal theory approach then allows

one to find an easy to use optimization procedure which often determines the action which is asymptotically optimal in a given state.

The third division of the paper deals with various specific systems. The general complicated nature of the problem of deciding the optimal policy is illustrated with the solution of a two component parallel system.

We then treat a series system in which the components have identical failure rates. The somewhat surprising conclusion is that the policy does not matter. The ergodic probability that the system works, the Laplace transform of the time to fix the system from any state and the expected integral of the discounted time the system works are all obtained.

For the series system with two components, the optimal policy is to repair the component with smaller failure rate first. This is true even if the repairman is subject to interruptions; intervals of time in which he is not allowed to work. This result shows that if the optimal policy of the series system is in the form of a list, the components must be repaired in order of increasing failure rate. A policy of such a form is intuitive for the series system, but no proof is yet available.

We next treat a system made of stochastically identical components. Often, symmetries in the structure function show that certain actions in certain states are non-optimal. Sometimes enough actions can be ruled out to

actually give the optimal policy. Several examples of the use of this technique are given.

1.2 Coherent Structure Theory

One of the common assumptions for reliability models is the existence of a coherent structure.[7] We include these assumptions for our model.

Specifically, we assume that the system is made up of n components, and that the system itself and each component can be in one of two states, functioning or failed. Thus the state of the n components can be summarized by a binary n-vector $\underline{X}$ with the understanding that $X_i = 1$ iff the $i^{\underline{th}}$ component is functioning, and that $X_i = 0$ iff the $i^{\underline{th}}$ component is failed. This vector $\underline{X}$ will be called the state of the system.

We further assume that whether the system is functioning or failed is a function only of the states of the n components, or equivalently, if χ represents the set of all binary n-vectors,

$$\exists\, \phi: \chi \rightarrow \{0,1\}\ ,\ \text{s.t.}\ ,$$

$\phi(\underline{X}) = 1$ iff the system is functioning when the system's state is $\underline{X}$,

$\phi(\underline{X}) = 0$ iff the system is failed when the the system's state is $\underline{X}$.

The function ϕ is called the structure function.

It is reasonable to assume that ϕ is monotone nondecreasing, corresponding to the intuitive notion that

repair of a component cannot cause system failure.

A component, i, is relevant if $\phi(1_i, \underline{X}) \neq \phi(0_i, \underline{X})$ for some $\underline{X}$. Here the notation

$$1_i, \underline{X} = (X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n) \quad ,$$

and

$$0_i, \underline{X} = (X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n) \quad .$$

Without loss of generality we need only consider relevant components.

A structure function that is non-decreasing and for which all components are relevant will be called coherent.

Given a coherent structure function ϕ, $\underline{X}$ is a path vector if $\phi(\underline{X}) = 1$. The corresponding path set is $C_1(\underline{X}) \overset{\text{def}}{=} \{i | X_i = 1\}$. A path set is minimal if no proper subset of the path set is a path set. It is easily seen that $\phi(\underline{X}) = 1$ if and only if some minimal path set is a subset of $C_1(\underline{X})$. Every component must be in at least one minimal path set.

Similarly a vector $\underline{X}$ is a cut vector if $\phi(\underline{X}) = 0$. The corresponding cut set is $C_0(\underline{X}) \overset{\text{def}}{=} \{i | X_i = 0\}$. A cut set is minimal if no proper subset of the cut set is a cut set. It is easily seen that $\phi(\underline{X}) = 0$ if and only if some minimal cut set is a subset of $C_0(\underline{X})$. Every component must be in at least one minimal cut set.

1.3 The Model

We deal with an n component system with a coherent structure function ϕ. We assume that the evolution of the state vector, $\underline{X}$, is governed by the following rules:

1. The $i^{\underline{th}}$ component functions for a random period of time exponentially distributed with rate $\mu_i > 0$. After such a period of time, it fails.

2. The $i^{\underline{th}}$ component, when failed and under repair, is repaired (goes from failed to functioning state) in a random period of time, exponentially distributed with rate $\lambda_i > 0$. When the $i^{\underline{th}}$ component is failed and not under repair it does not change state.

3. The component evolutions (given the component under repair) are independent of each other and the previous history of the system.

We also assume the following two rules for assignment of the repairman.

4. The component under repair may be changed arbitrarily quickly.

5. At most, one component may be under repair at a given time. Note that we have the option of leaving the repairman idle.

A policy will be defined to be any rule for assignment of the repairman. A policy is a stationary pure policy if the repairman is assigned by a deterministic function of state only. For a stationary pure policy $\pi, \pi(\underline{X})$ will denote the component under repair in state $\underline{X}$.

Define F to be the expected integral of the discounted time the system is working. Thus for $S>0$, π an arbitrary policy, and $\underline{X}(t)$ the system state $\underline{X}$ at time t,

$$F(\pi,S,\underline{X}_0) = E_\pi\left[\int_0^\infty e^{-St}\,\phi(\underline{X}(t))dt \,\middle|\, \underline{X}(0)=\underline{X}_0\right] . \tag{1.3.1}$$

Standard results[36] tell us that there exists a stationary pure policy π^*, such that

$$F(\pi^*,S,\underline{X}_0) = \sup_\pi F(\pi,S,\underline{X}_0) . \tag{1.3.2}$$

The same π^* satisfies (1.3.2) regardless of $\underline{X}_0$.

When the state space is finite, as in our model, a stationary pure policy π^* exists, such that (1.3.2) is true for all $\underline{X}_0$ and $S<S_0$ with $S_0>0$. Furthermore,

$$\lim_{S\to 0} SF(\pi^*,S,\underline{X}_0) = \sum_{\underline{X}:\phi(\underline{X})=1} e_{\pi^*}(\underline{X}) , \tag{1.3.3}$$

where $e_{\pi^*}(\underline{X})$ is the ergodic probability of the state $\underline{X}$ under policy π^*. Thus to find a policy which maximizes $F(\pi,S,\underline{X}_0)$ over π for small valves of S, we need only find a stationary pure policy which maximizes the ergodic probability that the system works. Our objective in the following paper will normally be to maximize the ergodic probability that the system works over the stationary pure policies possible.

1.4 Strict Optimality of Assignment of Repairman

In this section we prove the intuitive notion that the repairman should never be idle unless all components of the system are working.

Theorem 1.4 A. Let π' be a stationary pure policy in which the repairman is idle in at least one state which is not $\underline{1}$. Then at least one state $\underline{X}^* \neq \underline{1}$ with the repairman idle must be positive recurrent under the Markov chain which describes the system evolution under π'.

Proof: Clearly $\underline{0}$ is positive recurrent since all $\mu_i > 0$, and since there are a finite number of states.

If the theorem is not true, then some component j is under repair in the $\underline{0}$ state. Since $\lambda_j > 0$, we obtain $1_j,\underline{0}$ is positive recurrent. Repeating this argument n times tells us that $\underline{1}$ is positive recurrent.

But again $\mu_i > 0$ implies all states are positive recurrent, a contradiction. End of proof of Theorem 1.4 A.

Theorem 1.4 B. A stationary pure policy π' which leaves the repairman idle in some state not $\underline{1}$ does not maximize $F(\pi,S,\underline{X}_0)$ for any value of S or any $\underline{X}_0$.

Proof: We contradict the optimality of π' by comparing it with a derived policy π''. The evolution of the state under π', $\underline{X}'(\omega,t)$; and under π'', $\underline{X}''(\omega,t)$; will be mapped into the

same probability space in such a way that $\underline{X}''(\omega,t) \geq \underline{X}'(\omega,t)$, $\forall\ \omega,t$. Thus, by the monotonicity of ϕ, $\phi(\underline{X}''(\omega,t)) \geq \phi(\underline{X}'(\omega,t))$, $\forall\ \omega,t$. Furthermore we show that $\phi(\underline{X}''(\omega,t)) = 1$ and $\phi(\underline{X}'(\omega,t)) = 0$ on a set large enough to contradict the optimality of π'.

Let $(\Omega,\mathcal{F},P)$ support two processes $\underline{X}'$ and $\underline{Y}$ with the following joint distribution: Let $\underline{X}'$ evolve according to policy π' with $\underline{X}'(\omega,0) = \underline{X}_0$. Let δ be the first time that $\underline{X}'$ reaches the state $\underline{X}^*$ whose existence is quaranteed by Theorem 1.4 A, let τ be time spent in $\underline{X}^*$ before the first failure. Conditional on $\underline{X}'$, let $\underline{Y}$ describe the system evolution with $\underline{Y}(\omega,0) = \underline{0}$ under the following policy: before time τ repair the smallest numbered component in $C_0(\underline{X}^*)$ which is failed in $\underline{Y}$ (if none are failed, leave the repairman idle) and after time t let the repairman be idle.

Now define $\underline{X}''$ on $(\Omega,\mathcal{F},P)$ as follows:

$$\begin{aligned} \underline{X}''(t) &= \underline{X}'(t),\ 0 \leq t < \delta\ , \\ X''(t) &= \underline{X}'(t)\ \mathrm{V}\ \underline{\underline{\Delta}}(t-\delta)\underline{Y}(t-\delta),\ \delta \leq t < \infty\ , \end{aligned} \tag{1.4.1}$$

where $(\underline{X}\ \mathrm{V}\ \underline{Y})_i = \max(X_i,Y_i)$, and where $\underline{\underline{\Delta}}(t-\delta)$ is a diagonal matrix with $\Delta_{ii}(t_0-\delta) = 0$ if $X_i'(t) = 1$ for some $\delta \leq t \leq t_0$, $=$ 1 otherwise. My thanks to James Pitman who considerably shortened the original argument to this point.

It is clear that $\underline{X}''$ is the proper system evolution under a policy which uses policy π' until $\underline{X}^*$ is reached, then repairs the lowest numbered failed component until there is

a failure in $C_1(\underline{X}^*)$, and then repairs the component $\pi'(\underline{X}')$ (unless this component is working, in which case the repairman is idle). This is some non-stationary probabilistic policy.

From (1.4.1) $\underline{X}''(t) \geq \underline{X}'(t)$. We now show strict domination for each of two cases.

Case 1. $\phi(\underline{X}^*) = 0$. We note that there is non-zero probability that under π'', all components in $C_0(\underline{X}^*)$ are repaired in the interval $(\delta,\delta+\tau)$. This shows strict domination since $\phi(\underline{1}) = 1$ for a coherent structure.

Case 2. $\phi(\underline{X}^*) = 1$. Pick a component $j \varepsilon C_0(\underline{X}^*)$ and let a minimal cut set containing j be K. Such a minimal cut set can always be found for any component of a coherent structure. Let $K^* = K \cap C_1(\underline{X}^*)$.

We know that there is non-zero probability that only the components in K^* fail without repairs occuring for $\underline{X}'$ in an interval beginning at δ, while, during the same interval, all components in $C_0(\underline{X}^*)$ are repaired for $\underline{X}''$ without failures occuring for these components.

In this case $\phi(\underline{X}') = 0$ since K is a subset of the failed components of $\phi(\underline{X}')$, and $\phi(\underline{X}'') = 1$ since the failed components of $\underline{X}''$ are a proper subset of a minimal cut set $\underline{K}$. End of proof of Theorem 1.4 B.

Corollary: A stationary pure policy which does not assign the repairman in a state not $\underline{1}$ cannot achieve the maximum

ergodic probability that the system works.

<u>Definition</u>: A stationary pure policy which assigns the repairman in all states $\neq \underline{1}$ will be called a candidate policy. (1.4.2)

Thus, in optimizing the ergodic probability that the system works, we need only consider candidate policies.

Arguments similar to those used in the proof of Theorem 1.4 A tell us that the system evolution under a candidate policy is governed by a finite, irreducible continuous time Markov chain.

DIVISION II

Approximations and Bounds

CHAPTER 2

The Time Reversible Policy

2.1 Simultaneous Effort and Mixed Policies

Suppose that we allow the repairman to simultaneously expend a portion of his effort on several failed components. If, while in state $\underline{X}$, he expends a proportion of his effort $p_i(\underline{X})$ on the $i^{\underline{th}}$ failed component, it is reasonable to assume that the $i^{\underline{th}}$ component's repair rate is then $p_i(\underline{X})\lambda_i$, with failure rates remaining unchanged. By arguments similar to those advanced in section 1.4, if a policy is such that the repairman does not expend all his effort in a given state, the system evolution under that policy is dominated by the system evolution under another policy which expends all the repairman's effort in the same state. Also, standard results tell us that an optimal policy is a stationary policy.

Thus, in finding optimal policies when allowing the repairman the flexibility of simultaneous effort described in the previous paragraph, we need only consider candidate simultaneous effort policies defined below.

(2.1.1) Definition: A candidate simultaneous effort policy is described by a set of probability n-vectors $p(\underline{X})$, $\underline{X} \neq 1$, with the properties:

a) $p_i(\underline{X}) = 0$, $i \varepsilon C_1(\underline{X})$,

b) $p_i(\underline{X}) \geq 0$,

c) $\sum_{i=1}^{n} p_i(\underline{X}) = 1$,

and with the interpretation that the proportion of the

repairman's effort spent on component i in state $\underline{X}$ is $p_i(\underline{X})$. Thus, the repair rate of component i in state $\underline{X}$, $i \varepsilon C_0(\underline{X})$ is $p_i(\underline{X})\lambda_i$. The failure rate of component i is unchanged and is μ_i.

Note that a candidate policy is a special case of a candidate simultaneous effort policy.

Another model which gives rise to restricted simultaneous effort policies is the following: the system possesses m repair resources, or "men." If the $i^{\underline{th}}$ man is allocated to the $j^{\underline{th}}$ component, the repair rate for the $j^{\underline{th}}$ component is λ_{ij}. We assume there is no comparative advantage of men in repairing components, or more precisely, $\lambda_{i_1 j}/\lambda_{i_2 j}$ is independent of j. Furthermore, we assume that the application of repair resources to the system is linear, or more precisely, if the men in a set I_j are assigned to fix component j, then the repair rate of component j is $\sum_{i \varepsilon I_j} \lambda_{ij}$.

The fact that there is no comparative advantage of men in fixing components requires that

$$\lambda_{ij} = r_i \lambda_j \qquad (2.1.2)$$

where r_i can be normalized so that $\sum_{i=1}^{m} r_i = 1$.

Thus, in this model, if we employ a stationary policy wherein the men in the set $I_j(\underline{X})$ are assigned to repair failed component j in state $\underline{X}$, with $\bigcup_{j=1}^{n} I_j(\underline{X}) = \{1,2,...,m\}$, $\underline{X} \neq 1$, then the system evolution is equivalent

to the system evolution under a candidate simultaneous effort policy of definition (2.1.1), with

$$p_j(\underline{X}) = \sum_{i \varepsilon I_j(\underline{X})} r_i \quad , \qquad (2.1.3)$$

and λ_j and r_i given by (2.1.2).

(2.1.4) Definition. A policy, π, is called a stationary probabilistic or mixed policy, if upon change of the system state to $\underline{X}$, the repairman is assigned to repair component i with probability $q_i(\underline{X})$, and is not reassigned until the system changes state again.

The following theorem provides a way to compare ergodic probabilities under simultaneous effort policies with ergodic probabilities under mixed policies.

Theorem 2.1 A. Let $e(\underline{X})$ be the ergodic probability of state $\underline{X}$ under the Markov chain which describes the system evolution under a policy given in definition (2.1.1), and let $e'(\underline{X},i)$ be the ergodic probability of state $\underline{X}$ with the $i^{\underline{th}}$ component under repair in the Markov chain which describes the system evolution under definition (2.1.4). Furthermore let:

$$q_i(\underline{X}) = \begin{cases} p_i(\underline{X})(\lambda_i+\mu^*(\underline{X})) \Big/ \sum_{j \varepsilon C_0(\underline{X})} p_j(\underline{X})(\lambda_j+\mu^*(\underline{X})), & i \varepsilon C_0(\underline{X}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu^*(\underline{X}) = \sum_{i \varepsilon C_1(\underline{X})} \mu_i$, and $\underline{q}$ and $\underline{p}$ are the $\underline{q}$ and $\underline{p}$ from the appropriate definitions qiven above.

Then

$$e(\underline{X}) = \sum_{i \varepsilon C_0(\underline{X})} e'(\underline{X},i) \qquad (2.1.6)$$

Proof: All states of the second chain which are possible to enter are easily seen to be positive recurrent.

Since the first chain is ergodic, it satisfies the balance equations.

$$[\mu^*(\underline{X}) + \sum_{i \varepsilon C_0(\underline{X})} p_i(\underline{X})\lambda_i]e(\underline{X}) = \sum_{i \varepsilon C_1(\underline{X})} \lambda_i p_i(0_i,\underline{X})e(0_i,\underline{X}) + \sum_{i \varepsilon C_0(\underline{X})} \mu_i e(1_i,\underline{X}) \qquad (2.1.7)$$

We will show that

$$e'(\underline{X},i) = e(\underline{X})p_i(\underline{X}) \quad , \qquad (2.1.8)$$

which proves the desired result. This is done by showing that (2.1.8) satisfies the balance equations for the second chain, which are:

$$(\mu^*(\underline{X}) + \lambda_i)e'(\underline{X},i) = q_i(\underline{X})[\sum_{j \varepsilon C_1(\underline{X})} \lambda_j e'(0_j,\underline{X},j) + \sum_k \sum_{j \varepsilon C_0(\underline{X})} \mu_j e'(1_j,\underline{X},k)] \quad . \qquad (2.1.9)$$

Substituting (2.1.8) into (2.1.9), we obtain

$$p_i(\underline{X})(\mu^*(\underline{X})+\lambda_i)e(\underline{X}) = q_i(\underline{X})\Big[\sum_{j\varepsilon C_1(\underline{X})} \lambda_j p_j(0_j,\underline{X})e(0_j,\underline{X}) + \tag{2.1.10}$$

$$\sum_{j\varepsilon C_0(\underline{X})} \mu_j e(1_j,\underline{X})\Big] .$$

Substituting (2.1.7) into the square brackets gives:

$$p_i(\underline{X})(\mu^*(\underline{X})+\lambda_i)e(\underline{X}) = \tag{2.1.11}$$

$$q_i(\underline{X})\Big[(\mu^*(\underline{X}) + \sum_{j\varepsilon C_0(\underline{X})} p_j(\underline{X})\lambda_j)e(\underline{X})\Big]$$

and equality in (2.1.11) is guaranteed by (2.1.5).

We note that the transformation between $\underline{p}$ and $\underline{q}$ of (2.1.5) is a one to one and onto transformation between probability vectors defined to be non-zero on $C_0(\underline{X})$. End of proof of Theorem 2.1 A.

The inverse transformation of (2.1.5) is

$$p_i(\underline{X}) = \frac{\dfrac{q_i(\underline{X})}{\lambda_i+\mu^*(\underline{X})}}{\displaystyle\sum_{j\varepsilon C_0(\underline{X})} \frac{q_j(\underline{X})}{\lambda_j+\ ^*(\underline{X})}} \tag{2.1.12}$$

We have the following collary to Theorem 2.1 A:

If the repairman is allowed to expend simultaneous effort as in definition (2.1.1), the ergodic probability that the system works may be maximized by maximizing the ergodic probability that the system works over candidate policies. This follows from Theorem 2.1 A and the standard result that mixed policies need not be considered for optimization purposes.

Thus, for the model of linear repair resources without comparative advantage presented earlier, all resources may optimally be allocated at any time to one failed component.

2.2 The Time Reversible Chain Simultaneous Effort Policy

A continuous time Markov chain is said to be time reversible[24] if there is equality of ergodic flows between any two states. Thus we require $\lambda_{ij}e_i = \lambda_{ji}e_j$ for any two states i and j, where λ_{ij} is the rate of transitions from i to j, and e_i is the ergodic probability of state i.

Theorem 2.2 A. There is a unique set of probability vectors $\underline{p}(\underline{X})$, $\underline{X} \neq 1$ such that $p_i(\underline{X}) = 0$, $i \varepsilon C_1(\underline{X})$, for which the system evolution under the policy described in definition (2.1.1) is governed by a time reversible Markov chain. This policy will be called the time reversible policy.

Furthermore, the unique $\underline{p}(\underline{X})$ are determined for $\underline{X} \neq 1$ by

$$p_i(\underline{X}) = \begin{cases} \dfrac{1}{|C_0(\underline{X})|} & i\varepsilon C_0(\underline{X}) \\ 0 & \text{otherwise} \end{cases} \qquad (2.2.1)$$

and the ergodic potentials,[24] $\pi(\underline{X})$, are

$$\pi(\underline{X}) = |C_0(\underline{X})|\,! \prod_{i\varepsilon C_0(\underline{X})} \left(\frac{\mu_i}{\lambda_i}\right) . \qquad (2.2.2)$$

Here $|C_0(\underline{X})|$ denotes the cardinality of $C_0(\underline{X})$.

Proof: Assume the existence of the proper $\underline{p}(\underline{X})$. Arbitrarily define the potential of $\underline{1}$ to be 1,

$$\pi(\underline{1}) = 1 \quad . \qquad (2.2.3)$$

We now prove the theorem by induction on $|C_0(\underline{X})|$. When $|C_0(\underline{X})| = 1$, $\underline{p}(\underline{X})$ must be given by (2.2.1). Also by balance of ergodic flows between $0_i,\underline{1}$ and $\underline{1}$

$$\lambda_i \pi(0_i,\underline{1}) = \mu_i \pi(\underline{1}) \quad , \qquad \text{or} \qquad (2.2.4)$$

$$\pi(0_i,\underline{1}) = \frac{\mu_i}{\lambda_i} \quad , \qquad (2.2.5)$$

which satisfies the theorem.

Now assume that the theorem is true for $|C_0(\underline{X})| = m-1$. By equating ergodic flows, we obtain:

$$\lambda_{i_j} p_{i_j}(0_{i_1},\ldots 0_{i_m},\underline{1})\pi(0_{i_1},\ldots,0_{i_m},\underline{1}) =$$

$$\mu_{i_j}\pi(0_{i_1},\ldots,0_{i_{j-1}},0_{i_{j+1}},\ldots,0_{i_m},\underline{1}) \qquad (2.2.6)$$

$$\text{for } j = 1,\ldots,m \quad .$$

Now, using the induction hypothesis that (2.2.2) is true for the right hand side of (2.2.6), we obtain

$$p_{i_j}(0_{i_1},\ldots,0_{i_m},\underline{1})\pi(0_{i_1},\ldots,0_{i_m},\underline{1}) =$$

$$(m-1)\,!\,\prod_{j=1}^{m}\left(\frac{\mu_{i_j}}{\lambda_{i_j}}\right) \qquad (2.2.7)$$

or that all p_{i_j} are equal, $i_j \varepsilon C_0(\underline{X})$.

Thus,

$$p_{i_j}(0_{i_1},\ldots,0_{i_m},\underline{1}) = \frac{1}{m} , \tag{2.2.8}$$

and

$$\pi(0_{i_1},\ldots,0_{i_m},\underline{1}) = m! \prod_{j=1}^{m} \left(\frac{\mu_{i_j}}{\lambda_{i_j}}\right) . \tag{2.2.9}$$

The theorem is therefore true for $|C_0(\underline{X})| = m$, and the induction is complete.

We have actually constructed a time reversible chain, and assuming its existence, shown its uniqueness. End of proof of Theorem 2.2 A.

The ergodic probabilities are just the normalized potentials. Therefore the ergodic probability that the system works, W, under the time reversible policy is

$$W = \frac{\sum_{\underline{X}:\phi(\underline{X})=1} \pi(\underline{X})}{\sum_{\underline{X}} \pi(\underline{X})} \tag{2.2.10}$$

where $\pi(\underline{X})$ is given by (2.2.2).

W is a lower bound to the optimal ergodic reliability of the system.

Theorem 2.1 A tells us that W can be achieved by the mixed policy given in definition (2.1.4) i

$$q_i(\underline{X}) = \frac{\lambda_i+\mu^*(\underline{X})}{\sum_{j\epsilon C_0(\underline{X})} (\lambda_j+\mu^*(\underline{X}))} \qquad (2.2.11)$$

for $i\epsilon C_0(\underline{X})$.

Of course the system evolution in this case is not given by a time reversible chain.

2.3 Association of the Components Under the Ergodic Probabilities of the Time Reversible Policy

The bound, W, of the previous section is difficult to compute if the number of components in the system is substantial, since the number of states is equal to 2^n, where n is the number of components.

The computation of a bound for the ergodic probability that the system works could be simplified a great deal if the Esary-Proschan-Walkup cut set bound[16] is appropriate. This requires association of the components. It is conjectured that when the state probabilities are given by the normalized potentials of (2.2.2), the components are associated. However, only the association of any two components is proven in this section.

Definition:[7] Random variables $T_1,\ldots,T_n$ are associated if $\mathrm{cov}[\Gamma(\underline{T}),\Delta(\underline{T})] \geq 0$ for all pairs of increasing binary functions Γ,Δ.

Therefore, the components are associated if:

$$E[\Gamma(\underline{X})\Delta(\underline{X})] \geq E[\Gamma(\underline{X})]E[\Delta(\underline{X})] \tag{2.3.1}$$

for all increasing binary functions Γ,Δ. (2.3.1) is equivalent to

$$p[\Gamma(\underline{X})=1,\Delta(\underline{X})=1] \geq p[\Gamma(\underline{X})=1]p[\Delta(\underline{X})=1] \quad . \tag{2.3.2}$$

Therefore, using (2.2.2), we have that the coordinates are associated under the ergodic probabilities of the time reversible policy of:

$$\left[\sum_{\underline{X}} \pi(\underline{X})\right] \left[\sum_{\substack{\underline{X}:\Gamma(\underline{X})=1\\ \Delta(\underline{X})=1}} \pi(\underline{X})\right] \geq \left[\sum_{\underline{X}:\Gamma(\underline{X})=1} \pi(\underline{X})\right] \left[\sum_{\underline{X}:\Delta(\underline{X})=1} \pi(\underline{X})\right], \tag{2.3.3}$$

for all increasing binary functions Γ and Δ. (2.3.3) is equivalent to

$$\left[\sum_{\substack{\underline{X}:\Gamma(\underline{X})=0\\ \Delta(\underline{X})=0}} \pi(\underline{X})\right] \left[\sum_{\substack{\underline{X}:\Gamma(\underline{X})=1\\ \Delta(\underline{X})=1}} \pi(\underline{X})\right] \geq \left[\sum_{\substack{\underline{X}:\Gamma(\underline{X})=1\\ \Delta(\underline{X})=0}} \pi(\underline{X})\right] \left[\sum_{\substack{\underline{X}:\Delta(\underline{X})=1\\ \Gamma(\underline{X})=0}} \pi(\underline{X})\right] \tag{2.3.4}$$

Theorem 2.3 A. When $\pi(\underline{X})$ is given by (2.2.2), (2.3.4) is true for all binary increasing functions Γ and Δ of the coordinates X_i and X_j.

Proof: If the right hand side of (2.3.4) is zero, the inequality is satisfied, since all terms on the left hand

side are positive.

The only binary increasing Γ and Δ which give a non-zero right hand side are $\Gamma = X_i$, $\Delta = X_j$ or vice versa. Thus we need show that

$$\left(\sum_{\substack{\underline{X}:X_i=0\\ X_j=0}} \pi(\underline{X})\right)\left(\sum_{\substack{\underline{X}:X_i=1\\ X_j=1}} \pi(\underline{X})\right) \geq \left(\sum_{\substack{\underline{X}:X_i=1\\ X_j=0}} \pi(\underline{X})\right)\left(\sum_{\substack{\underline{X}:X_i=0\\ X_j=1}} \pi(\underline{X})\right) \tag{2.3.5}$$

We accomplish this through the inequality

$$\pi(0_i,0_j,\underline{X})\pi(1_i,1_j,\underline{Y}) + \pi(0_i,0_j,\underline{Y})\pi(1_i,1_j,\underline{X}) \geq \pi(0_i,1_j,\underline{X})\pi(1_i,0_j,\underline{Y}) + \pi(0_i,1_j,\underline{Y})\pi(1_i,0_j,\underline{X}) \tag{2.3.6}$$

When (2.3.6) is summed over all $\underline{X}$ and $\underline{Y}$ we obtain 32 times (2.3.5).

To prove (2.3.6) let $a = |C_0(1_i,1_j,\underline{X})|$

$b = |C_0(1_i,1_j,\underline{Y})|$.

Then by (2.2.2), (2.3.6) reduces, after division by a common factor, to

$$a!(b+2)! + b!(a+2)! \geq 2(a+1)!(b+1)! \tag{2.3.7}$$

which is equivalent to

$$\frac{b+2}{a+1} + \frac{a+2}{b+1} \geq 2 \qquad (2.3.8)$$

which is readily verified to be true for $a,b \geq 0$. End of proof of Theorem 2.3 A.

Actually, (2.3.4) seems to be true for Γ and Δ arbitrary binary increasing functions of all coordinates, but the proof seems rather involved.

CHAPTER 3

Highly Reliable Systems

Approximations of highly reliable systems are treated in the following chapter. We assume that the $i^{\underline{th}}$ component's failure rate is $k\mu_i$ and the $i^{\underline{th}}$ component's repair rate is λ_i. As k goes to zero, we wish to know the optimal policy, and also the approximate optimal unreliability of the system. An example of the application of the results obtained in this chapter to a highly reliable system is given.

3.1 Limiting Ergodic Probability of States for Candidate Policies

Theorem 3.1 A. Let the failure rate of the $i^{\underline{th}}$ component be $k\mu_i$, and the repair rate of the $i^{\underline{th}}$ component be λ_i. Let π be a candidate policy (see section 1.4), and $e_\pi(\underline{X},k)$ be the ergodic probability of state $\underline{X}$ under policy π. Then

$$\lim_{k\to 0} (k^{1-|C_0(\underline{X})|})\, e_\pi(\underline{X},k) = 0 \quad . \tag{3.1.1}$$

Proof: The result is proven by induction on $|C_0(\underline{X})|$.

Clearly (3.1.1) holds for $|C_0(\underline{X})| = 0$, since $0 \leq e_\pi(\underline{X},k) \leq 1$. Now assume (3.1.1) is true for $|C_0(\underline{X})| = 0,\ldots,m<n$.

When we equate the ergodic rate of entering and leaving the group of states for which $|C_0(\underline{X})| \leq m$, we obtain

$$\sum_{\underline{X}:|C_0(\underline{X})|=m+1} \lambda_{\pi(\underline{X})} e_\pi(\underline{X},k) = \sum_{X:|C_0(\underline{X})|=m} e_\pi(\underline{X},k) \sum_{i\varepsilon C_1(\underline{X})} k\mu_i \tag{3.1.2}$$

Multiply both sides of equation (3.1.2) by k^{-m} and take the limit as k goes to zero. By the induction hypothesis, the right hand side has a limit of zero as k goes to zero.

Since the left hand side is a sum of non-negative

terms, the limit of each term as $k \to 0$ must be zero. The induction is established since $\lambda_{\pi(\underline{X})} \neq 0$, $\forall$ $\underline{X}$ s.t. $|C_0(\underline{X})| = m + 1 > 0$. End of proof of Theorem 3.1 A.

For the remainder of the chapter, we suppress the dependence of e_π on k.

<u>Theorem 3.1 B</u>. Under the conditions of Theorem 3.1 A,

$$\lim_{k\to 0} k^{-|C_0(\underline{X})|} e_\pi(\underline{X}) = g_\pi(\underline{X}) \quad , \qquad (3.1.3)$$

with $0 < g_\pi(\underline{X}) < \infty$, and $g_\pi(\underline{1}) = 1$,

$$\lambda_{\pi(\underline{X})} g_\pi(\underline{X}) = \sum_{i\varepsilon C_0(\underline{X})} \mu_i g_\pi(1_i,\underline{X}) \quad , \quad \underline{X} \neq \underline{1} \quad . \qquad (3.1.4)$$

Thus, $g_\pi(\underline{X})$ can be computed recursively for increasing $|C_0(\underline{X})|$.

<u>Proof</u>: Again use induction on $|C_0(\underline{X})|$. We note that $\sum_{\underline{X}} e_\pi(\underline{X}) = 1$. Theorem 3.1 A tells us that $\lim_{k\to 0} e_\pi(\underline{X}) = 0$, $\underline{X} \neq \underline{1}$. Thus, $\lim_{k\to 0} e_\pi(\underline{1}) = 1$, establishing the theorem for $|C_0(\underline{X})| = 0$.

Now assume that the theorem is true for all $\underline{X}$ such that $|C_0(\underline{X})| = m$.

Equating the ergodic rate of leaving and entering a state $\underline{X}$ yields:

$$(\lambda_\pi(\underline{X}) + k \sum_{i\epsilon C_1(\underline{X})} \mu_i) e_\pi(\underline{X}) =$$

$$\sum_{\substack{i\epsilon C_1(\underline{X}) \\ i=\pi(0_i,\underline{X})}} \lambda_i e_\pi(0_i,\underline{X}) + k \sum_{i\epsilon C_0(\underline{X})} \mu_i e_\pi(1_i,\underline{X}) \quad . \tag{3.1.5}$$

Assume $|C_0(\underline{X})| = m+1$. Multiply both sides of (3.1.5) by $k^{-(m+1)}$. Since $i\epsilon C_1(\underline{X})$ implies $|C_0(0_i,\underline{X})| = m+2$, Theorem 3.1 A tells us all terms in the first summation on the right hand side have zero limit.

Also $i\epsilon C_0(\underline{X})$ implies that $|C_0(1_i,\underline{X})| = m$, so that the induction hypothesis tells us the limit of the second sum on the right hand side is $\sum_{i\epsilon C_0(\underline{X})} \mu_i g_\pi(\underline{X})$.

Theorem 3.1 A tells us the second term on the left hand size has zero limit. We conclude that

$$\lim_{k\to 0} k^{-(m+1)} \lambda_{\pi(\underline{X})} e_\pi(\underline{X}) = \sum_{i\epsilon C_0(\underline{X})} \mu_i g_\pi(1_i,\underline{X}) \quad ,$$

thus establishing (3.1.3) and (3.1.4) for $|C_0(\underline{X})| = m+1$.

The fact that $0 < g_\pi(\underline{X}) < \infty$ is established by the induction hypothesis, (3.1.4), and the fact that the right hand summation of (3.1.4) is non-empty for $\underline{X} \neq \underline{1}$.

We are now interested in higher order approximations to the probability that the system works.

<u>Theorem 3.1 C</u>. Under the conditions of Theorem

3.1 A,

$$\lim_{k\to 0} k^{-|C_0(\underline{X})|-1}(e_\pi(\underline{X})-k^{|C_0(\underline{X})|}g_\pi(\underline{X})) = g_\pi^{(2)}(\underline{X}) \tag{3.1.6}$$

where $g_\pi^{(2)}(\underline{X})$ can be determined recursively on $|C_0(\underline{X})|$ by the equations

$$g_\pi^{(2)}(\underline{1}) = -\sum_{i=1}^{n} g_\pi(0_i,\underline{1}) \quad , \quad \text{and}$$

$$\lambda_{\pi(\underline{X})} g_\pi^{(2)}(\underline{X}) = \sum_{\substack{i\varepsilon C_1(\underline{X}) \\ i=\pi(0_i,\underline{X})}} \lambda_i g_\pi(0_i,\underline{X}) + \tag{3.1.7}$$

$$\sum_{i\varepsilon C_0(\underline{X})} \mu_i g_\pi^{(2)}(1_i,\underline{X}) - \sum_{i\varepsilon C_1(\underline{X})} \mu_i g_\pi(\underline{X}) \quad , \quad \underline{X} \neq \underline{1} \, .$$

<u>Proof</u>: Equations (3.1.4) and (3.1.5) can be combined to give:

$$\lambda_{\pi(\underline{X})} [e_\pi(\underline{X})-k^{|C_0(\underline{X})|}g_\pi(\underline{X})] = \sum_{\substack{i\varepsilon C_1(\underline{X}) \\ i=\pi(0_i,\underline{X})}} \lambda_i e_\pi(0_i,\underline{X}) +$$

$$k \sum_{i\varepsilon C_0(\underline{X})} \mu_i [e_\pi(1_i,\underline{X})-k^{|C_0(1_i,\underline{X})|}g_\pi(1_i,\underline{X})] -$$

$$k \sum_{i\varepsilon C_1(\underline{X})} \mu_i e_\pi(\underline{X}) \quad , \quad \underline{X} \neq \underline{1} \, . \tag{3.1.8}$$

We prove the theorem by induction on $C_0(\underline{X})$.

When $\underline{X} = \underline{1}$ use the fact that $\sum_{\underline{X}} e_\pi(\underline{X}) = 1$ and theorem 3.1 B to establish the result.

Now assume the theorem is true for $\underline{X}$ such that $|C_0(\underline{X})| = m$. We wish to establish that the theorem is true for $\underline{X}$ such that $|C_0(\underline{X})| = m+1$.

Multiply both sides of (3.1.8) by $k^{-|C_0(\underline{X})|-1}$ and take the limit as k goes to zero. Theorem 3.1 B and the induction hypothesis then give the desired result for $\underline{X}$ such that $|C_0(\underline{X})| = m+1$. End of proof of the theorem.

In a similar manner one can prove the following theorem.

<u>Theorem 3.1 D</u>. Under the conditions of theorem 3.1 A, define $g_\pi^{(1)}(\underline{X}) = g_\pi(\underline{X})$, then

$$\lim_{k\to 0} k^{-|C_0(\underline{X})|-m}[e_\pi(\underline{X}) - k^{|C_0(\underline{X})|} g_\pi^{(1)}(\underline{X}) - \ldots - k^{|C_0(\underline{X})|+m-1} g_\pi^{(m)}(\underline{X})]$$

$$= g_\pi^{(m+1)}(\underline{X}) \qquad (3.1.9)$$

Where $g_\pi^{(m+1)}(\underline{X})$ can be determined recursively in terms of smaller m and $|C_0(\underline{X})|$ as follows

$$g_\pi^{(m+1)}(\underline{1}) = - \sum_{\underline{X}:0<|C_0(\underline{X})|<m+1} g_\pi^{(m+1-|C_0(\underline{X})|)}(\underline{X})$$

and

$$\lambda_{\pi(\underline{X})} g_{\pi}^{(m+1)}(\underline{X}) = \sum_{\substack{i \varepsilon C_1(\underline{X}) \\ i=\pi(0_i,\underline{X})}} \lambda_i g_{\pi}^{(m)}(0_i,\underline{X}) + \tag{3.1.10}$$

$$\sum_{i \varepsilon C_0(\underline{X})} \mu_i g_{\pi}^{(m+1)}(1_i,\underline{X}) - \sum_{i \varepsilon C_1(\underline{X})} \mu_i g_{\pi}^{(m)}(\underline{X}) \quad ,$$

$$\underline{X} \neq \underline{1} \quad .$$

Proof: The proof exactly parallels the proof of theorem 3.1 C and is omitted.

3.2 Limiting Optimal Ergodic Unreliability of the System

Theorem 3.2 A. For a given $\underline{X}$, a candidate policy π minimizes $g_\pi(\underline{X})$ given in theorem 3.1 B over possible candidate policies if and only if $\lambda_{\pi(\underline{Y})} \geq \lambda_i \; \forall_i \varepsilon C_0(\underline{Y})$ for all $\underline{Y}$ such that $C_0(\underline{Y}) \subset C_0(\underline{X})$.

Furthermore,

$$\min_\pi g_\pi(\underline{X}) = \left[\prod_{i \varepsilon C_0(\underline{X})} \mu_i\right]\left[\sum_{\substack{\text{permutations}\\ \text{of } C_0(\underline{X})}} \prod_{j=1}^{|C_0(\underline{X})|} \frac{1}{\lambda_j^*}\right] \qquad (3.2.1)$$

where $\lambda_j^* = \max_{\substack{k \,\varepsilon\, \text{first } j\\ \text{elements of}\\ \text{the permutation}}} \lambda_k$.

Proof: Again by induction on $|C_0(\underline{X})|$. The theorem is true vacuously when $|C_0(\underline{X})| = 0$ or $|C_0(\underline{X})| = 1$. Now assume the theorem is true for all $\underline{Y}$ such that $|C_0(\underline{Y})| = m$.

Let $\underline{X}$ be such that $|C_0(\underline{X})| = m+1$. Equation (3.1.4) tells us that $g_\pi(\underline{X})$ is minimized iff $\lambda_{\pi(\underline{X})} \geq \lambda_i$ $i\varepsilon C_0(\underline{X})$, and $g_\pi(1_i,\underline{X})$ is minimized for all $i\varepsilon C_0(\underline{X})$.

Therefore, by the induction hypothesis, $g_\pi(\underline{X})$ is minimized iff $\lambda_{\pi(\underline{Y})} \geq \lambda_i \forall i \varepsilon C_0(\underline{Y})$ for all $\underline{Y}$ s.t. $C_0(\underline{Y}) \subset C_0(\underline{X})$; establishing the first part of the theorem.

For the second part, note that (3.1.4) tells us

$$\min_{\pi} g_{\pi}(\underline{X}) = \frac{1}{\max\limits_{j\varepsilon C_0(\underline{X})} \lambda_j} \sum_{i\varepsilon C_0(\underline{X})} \mu_i \min_{\pi} g_{\pi}(1_i,\underline{X}) \quad . \qquad (3.2.2)$$

Substituting for $\min\limits_{\pi} g_{\pi}(1_i,\underline{X})$ by the induction hypothesis gives

$$\frac{\min\limits_{\pi} g_{\pi}(\underline{X})}{\prod\limits_{i\varepsilon C_0(\underline{X})} \mu_i} = \frac{1}{\max\limits_{j\varepsilon C_0(\underline{X})} \lambda_j} \sum_{i\varepsilon C_0(\underline{X})} \sum_{\substack{\text{permutations}\\ \text{of } C_0(\underline{X})-\{i\}}} \prod_{j=1}^{|C_0(\underline{X})|-1} \frac{1}{\lambda_j^*} \quad . \qquad (3.2.3)$$

but the double sum simplifies to give:

$$\sum_{\substack{\text{permutations}\\ \text{of } C_0(\underline{X})}} \prod_{j=1}^{|C_0(\underline{X})|-1} \frac{1}{\lambda_j^*} \quad ,$$

and we also obtain by definition

$$\max_{j\varepsilon C_0(\underline{X})} \lambda_j = \lambda^*_{|C_0(\underline{X})|}$$

for any permutation, and thus the second result is established. End of proof of theorem 3.2 A.

Theorem 3.2 B. Let $U(k)$ be the optimal ergodic unreliability of the previously described system. Let m_1 = min cardinality of the cut sets = min cardinality of the minimal cut sets. Then

$$\lim_{k\to 0} k^{-m_1} U(k) = C_1 \qquad (3.2.4)$$

where C_1 is defined below.

$$C_1 = \sum_{\substack{K: \\ K \text{ is a cut set} \\ |K| = m_1}} \prod_K \mu_i \sum_{\substack{\text{permutations} \\ \text{of } K}} \prod_{j=1}^{m_1} \frac{1}{\lambda_{j*}} \tag{3.2.5}$$

where $\lambda_{j*} = \max_{i \in \text{ first } j \text{ elements of the permutation}} \lambda_i$.

Furthermore, for all $k < k_0$, for some $k_0 > 0$, a policy π is not optimal unless, for every cut set K with $|K| = m_1$ and every $\underline{Y}$ such that $C_0(\underline{Y}) \subset K$, $\lambda_{\pi(\underline{Y})} \geq \lambda_i \forall i \epsilon C_0(\underline{Y})$.

<u>Proof</u>: Let π^0 be a candidate policy which minimizes all $g_\pi(\underline{X})$ as given in theorem 3.2 A. Let π' be a candidate policy which does not have the property given in the last paragraph of theorem 3.2 B. Let $U^0(K)$ be the ergodic unreliability of policy π^0, $U'(K)$ be the ergodic unreliability of policy π'. Then by theorem 3.2 A

$$\lim_{k \to 0} k^{-m_1} U^0(k) < \lim_{k \to 0} k^{-m_1} U'(k) \tag{3.2.6}$$

and thus for all $k < k(\pi')$ with $k(\pi') > 0$, policy π' is not optimal. Since the number of candidate policies is finite, the last paragraph of the theorem holds.

Furthermore, if π'' is a candidate policy which minimizes $g_\pi(\underline{X})$ for all $C_0(\underline{X})$ which are cut sets of cardinality m_1, then

$$\lim_{k \to 0} k^{-m_1} U''(k) = C_1 \qquad (3.2.7)$$

where C_1 is given by (3.2.5).

Thus (3.2.4) is true since for all $k < k_0$ the optimal policy must be of the form given for π'', of which there are a finite number. End of proof of theorem 3.2 B.

The higher order approximations of theorem 3.1 D can now be applied to further narrow the possible optimal policies for small k. Such extensions appear quite messy.

It appears somewhat easier to use alternating renewal theory to further specify the optimal policy for small k. Such an approach is taken in section 3.3 which follows.

3.3 Limiting Expected Passage Times

Theorem 3.3 A. For the previously described system, and π a candidate policy, let $T_\pi(k,\underline{X})$ be the expected time to go from state $\underline{X}$ under policy π to the $\underline{1}$ state. Then

$$\lim_{k\to 0} T_\pi(k,\underline{X}) = T(0,\underline{X}) = \sum_{i\in C_0(\underline{X})} \frac{1}{\lambda_i} < \infty \quad . \tag{3.3.1}$$

Proof: The equations for the expected passage times yield solutions continuous in k, and the passage time in the absence of failures is easy to compute and is independent of policy. End of proof of Theorem 3.3 A.

Theorem 3.3 B. Let $B_\pi(k,\underline{X})$ be the expected time the system is failed during passage from $\underline{X}$ to $\underline{1}$ under policy π. Let $I(\underline{X})$ be the minimum number of component failures in $\underline{X}$ to cause system failure. Then

$$\lim_{k\to 0} k^{1-I(\underline{X})} B_\pi(k,\underline{X}) = 0 \quad . \tag{3.3.2}$$

Proof: The proof is by induction on $I(\underline{X})$. Theorem 3.3 A establishes the result when $I(\underline{X}) = 0$.

Now assume that the theorem is true for all $\underline{X}$ such that $I(\underline{X}) = 0,1,\ldots,m$ and that

$$\lim_{k \to 0} k^{1-m} B_\pi(k,\underline{X}) = 0 \qquad (3.3.3)$$

for $\underline{X}$ such that $I(\underline{X}) \geq m$.

The following identity for $B_\pi(k,\underline{X})$, $\underline{X} \neq 1$, obtained by conditioning is helpful.

$$(\lambda_{\pi(\underline{X})} + k \sum_{i \varepsilon C_i(\underline{X})} \mu_i) B_\pi(k,\underline{X}) = \qquad (3.3.4)$$

$$\delta_{0\phi(\underline{X})} + \lambda_{\pi(\underline{X})} B_\pi(k,1_{\pi(\underline{X})},\underline{X}) + k \sum_{i \varepsilon C_1(\underline{X})} \mu_i B_\pi(k,0_i,\underline{X}) \quad .$$

Here δ is the Kronecker delta, $\delta_{ij} = 0$, $i \neq j$; $\delta_{ij} = 1$, $i = j$.

Now let $\underline{X}$ be such that $I(\underline{X}) \geq m+1$, which implies $\phi(\underline{X}) = 1$.

We must show that

$$\lim_{k \to 0} k^{-m} B_\pi(k,\underline{X}) = 0 \quad , \qquad (3.3.5)$$

and our inductive proof in complete.

Multiply both sides of (3.3.4) by k^{-m}, and let $k \to 0$. By (3.3.3) the sum on the left hand side goes to zero as k goes to zero. We also have that $\delta_{0\phi(\underline{X})} = 0$. The last term on the right hand side has zero limit since $I(0_i,\underline{X}) \geq I(\underline{X})-1 = m$, and (3.3.3). We will thus establish (3.3.5) as desired if

$$\lim_{k \to 0} k^{-m} B_\pi(k, 1_{\pi(\underline{X})}, \underline{X}) = 0 \quad . \tag{3.3.6}$$

Note that $I(1_{\pi(\underline{X})}, \underline{X}) \geq I(\underline{X}) = m+1$ so that (3.3.6) is the same type of statement as (3.3.5). We prove both of these by subinduction on $|C_0(\underline{X})|$.

$B_\pi(k, \underline{1}) = 0$ so that (3.3.5) is true when $C_0(\underline{X}) = 1$. If (3.3.5) is true for $|C_0(\underline{X})| = j$ then (3.3.6) is true for $|C_0(1_{\pi(\underline{X})}, \underline{X})| = j$ implying (3.3.5) is true for $|C_0(\underline{X})| = j+1$. End of the subinduction and end of inductive proof of Theorem 3.3 B.

<u>Theorem 3.3 C</u>.

$$\lim_{k \to 0} k^{-I(\underline{X})} B_\pi(k, \underline{X}) = d_\pi(\underline{X}) \tag{3.3.7}$$

where $0 < d_\pi(\underline{X}) < \infty$, $\underline{X} \neq \underline{1}$.

If $\phi(\underline{X}) = 1$, $d_\pi(\underline{X})$ are determined by

$$\lambda_{\pi(\underline{X})} d_{\pi(\underline{X})} = \delta_{I(\underline{X}), I(1_{\pi(\underline{X})}, \underline{X})} \lambda_{\pi(\underline{X})} d_\pi(1_{\pi(\underline{X})}, \underline{X}) + \tag{3.3.8}$$

$$\sum_{\substack{i \varepsilon C_1(\underline{X}) \\ I(0_i, \underline{X}) = I(\underline{X}) - 1}} \mu_i d_\pi(0_i, \underline{X})$$

If $\phi(\underline{X}) = 0$, $d_\pi(\underline{X})$ is the time to fix the system when $k = 0$.

<u>Proof</u>: The theorem is proven for $\phi(\underline{X}) = 0$ by a continuity

argument similar to that used in proving 3.3 A.

Now assume the theorem is true for $\underline{Y}$ such that $I(\underline{Y}) = 1,\ldots,m$. Let $I(\underline{X}) = m+1$. Multiply both sides of (3.3.4) by $k^{-(m+1)}$ and let $k \to 0$. The sum on the left hand side goes to zero by Theorem 3.3 B. The first term on the right hand side is zero, and the third term on the right hand side has a limit equal to the last term in (3.3.8) by the induction hypothesis, and Theorem 3.3 B. If $I(1_{\pi(\underline{X})},\underline{X}) > I(\underline{X})$ the second term on the right hand side has zero limit.

(3.3.8) is thus established for a maximal element of $\{\underline{X} | I(\underline{X}) = m+1\}$, and for a maximal element of the set deleting a maximal element, and so forth.

Inductively, $d_\pi(\underline{X}) < \infty$ and does not equal zero, since the last term on the right hand side of 3.3.8 is non-zero inductively.

3.4 Limiting Optimal State Actions Under First Order Passage Times

In an intuitive sense, we wish to minimize the expected time in failed states during passage between two states, as k goes to zero, since the expected passage time becomes independent of policy. We can make this idea more precise using alternating renewal theory.

Theorem 3.4 A. Suppose that for two candidate policies π^0 and π^1

$$\frac{B_{\pi^0}(k,\underline{X})}{T_{\pi^0}(k,\underline{X})} \geq b \text{ (ergodic probability the system is failed under } \pi'\text{)} \quad \text{for } k < k_0 \text{ with } k_0 > 0,\ b > 0 \tag{3.4.1}$$

and

$$B_{\pi^0}(k,\underline{X}) < c\, B_{\pi'}(k,\underline{X}) \quad \text{for } k < k_0 \text{, with } 0 \leq c < 1,\ k_0 > 0 \ . \tag{3.4.2}$$

Then as k goes to zero, π' is not optimal.

Proof: We contradict the optimality of π' as follows. Construct a policy π'' which uses policy π^0 between $\underline{X}$ and $\underline{1}$ and uses policy π' between $\underline{1}$ and $\underline{X}$.

Let $S(k,\underline{X})$ be the expected time to pass from $\underline{1}$ to $\underline{X}$ under policy π', and let $F(k,\underline{X})$ be the expected time the system is failed during passage from $\underline{1}$ to $\underline{X}$ under π'. By alternating renewal theory,[36,37] the ergodic probability that the system is failed under π' is

$$P_{\pi'}(\phi(\underline{X})=0) = \frac{B_{\pi'}(k,\underline{X})+F(k,\underline{X})}{T_{\pi'}(k,\underline{X})+S(k,\underline{X})} \tag{3.4.3}$$

and the ergodic probability that the system is failed under π'' is

$$P_{\pi''}(\phi(\underline{X})=0) = \frac{B_{\pi 0}(k,\underline{X})+F(k,\underline{X})}{T_{\pi 0}(k,\underline{X})+S(k,\underline{X})} \quad . \tag{3.4.4}$$

Now suppose π'' is not strictly better than π' or that

$$P_{\pi''}(\phi(\underline{X})=0) \geq P_{\pi'}(\phi(\underline{X})=0) \quad . \tag{3.4.5}$$

Substitutions from (3.4.3) and (3.4.4) into (3.4.5) and suppression of the arguments gives

$$T_{\pi 0} \leq T_{\pi'} + \frac{(T_{\pi'}+S)(B_{\pi 0}-B_{\pi'})}{(B_{\pi'}+F)} \quad , \tag{3.4.6}$$

using (3.4.2) in (3.4.6) gives

$$T_{\pi 0} \leq T_{\pi'} - (\frac{1}{c} - 1)B_{\pi 0}\frac{(T_{\pi'}+S)}{B_{\pi'}+F} \tag{3.4.7}$$

for $k < k_0$.

Using (3.4.1) and (3.4.3) in (3.4.7) we obtain

$$T_{\pi 0} \leq T_{\pi'} - (\frac{1}{c} - 1)bT_{\pi 0} \quad (3.4.8)$$

for $k < k_0$, a contradiction as $k \to 0$ since $\lim_{k \to 0} T_{\pi 0} = \lim_{k \to 0} T_{\pi'}$, by Theorem 3.3 A. End of proof of Theorem 3.4 A.

Theorem 3.4 B. A candidate policy π' which does not minimize the $d_\pi(\underline{X})$ of Theorem 3.3 C is not optimal for $k < k'$ for some $k' > 0$.

Proof: Suppose $d_{\pi''}(\underline{X}) < d_{\pi'}(\underline{X})$. Then by (3.3.7) of Theorem 3.3 C

$$B_{\pi''}(k,\underline{X}) < cB_{\pi'}(k,\underline{X}) \quad \text{for } c < 1, \text{ all } k < k_0,$$

satisfying (3.4.2) of Theorem 3.4 A.

By Theorem 3.2 B, if π' is optimal as k goes to zero then

$$\lim_{k \to 0} k^{-m_1} (\text{ergodic probability of failure under } \pi') = C_1 \quad (3.4.9)$$

where m_1 is the minimal size of a cut.

Theorem 3.3 C tells us that

$$\lim_{k \to 0} k^{-I(\underline{X})} B_{\pi''}(k,\underline{X}) = d_{\pi''}(\underline{X}) \quad . \quad (3.4.10)$$

Since $I(\underline{X}) \leq m_1$, (3.4.9), (3.4.10), and Theorem 3.3 A imply condition (3.4.1) of Theorem 3.4 A. Theorem 3.4 A now gives the desired result. End of proof.

We now use Theorem 3.4 B and equation (3.3.8) to find many of the optimal actions for the limiting optimal policy as k goes to zero.

When $\phi(\underline{X}) = 0$, minimize $\sum_{i \in A} \frac{1}{\lambda_i}$ over sets of components A which, when fixed in state $\underline{X}$, fix the system. One of these components must be under repair in the limiting optimal policy. Thus rule translates to: a failed system should be repaired in such a manner that the time to repair in the absence of failures is minimized.

Now successively minimize $d_{\pi(\underline{X})}$ in the following manner. First compute d_π for the set of $\underline{X}$ with $I(\underline{X}) = 1$. Choose a maximal element in this set and minimize according to (3.3.8). (The optimal action is to repair the failed component with largest λ_i in this case.) Continue to choose maximal elements of the uncomputed states with $I(\underline{X}) = 1$ minimizing according to (3.3.8) until the minimal d_π's are computed for all states $\underline{X}$, with $I(\underline{X}) = 1$. Now do the same procedure with the set of states with $I(\underline{X}) = 2$, then with the set of states with $I(\underline{X}) = 3$, and so forth until d is minimized for all states.

One easy example to compute is the parallel system. The limiting optimal policy for small k must always choose the largest repair rate failed component in any state.

CHAPTER 4

Highly Unreliable Systems

To model highly unreliable systems we may take the model of the last chapter and examine the behavior as k goes to infinity.

By a time rescaling argument such a model is equivalent to a system for which the i^{th} component's failure rate is μ_i and for which the i^{th} component's repair rate is $\frac{\lambda_i}{k}$.

Thus, we may equivalently consider a model for which the i^{th} component's failure rate is μ_i, and for which the i^{th} component's repair rate is $k\lambda_i$, and examine the behavior as k goes to zero.

Much of the analysis is similar to that of the last chapter, and we shall be interested in the approximate optimal reliability of the system and the optimal action for small k.

Intuitively the results for highly unreliable systems are of less usefulness than the results for highly reliable systems.

4.1 Limiting Ergodic Probability of States for Candidate Policies

For a candidate policy π (see section 1.4), define

$$L_\pi(\underline{X}) = \text{minimum possible number of repairs under } \pi \text{ to reach } \underline{X} \text{ from } \underline{0} \text{ irrespective of failures.} \tag{4.1.1}$$

Thus, $\underline{X} = \underline{0}$ is the only state $\underline{X}$ such that $L_\pi(\underline{X}) = 0$, and $\underline{X} = 1_{\pi(\underline{0})}, \underline{0}$ is the only state $\underline{X}$ such that $L_\pi(\underline{X}) = 1$. The only states $\underline{X}$ with $L_\pi(\underline{X}) = 2$ are $\underline{X} = 1_{\pi(\underline{0})}, 1_{(1_{\pi(\underline{0})}, \underline{0})}, \underline{0}$ and $\underline{X} = 1_{\pi(1_{\pi(\underline{0})}, \underline{0})}, \underline{0}$.

Theorem 4.1 A. $L_\pi(\underline{X})$ has the following properties:

$$L_\pi(\underline{X}) \leq n \quad . \tag{4.1.2}$$

$$L_\pi(\underline{X}) \geq |C_1(\underline{X})| \quad . \tag{4.1.3}$$

$$\text{If } \underline{X} \geq \underline{Y}, \text{ then } L_\pi(\underline{X}) \geq L_\pi(\underline{Y}) \quad . \tag{4.1.4}$$

$$\text{If } L_\pi(\underline{X}) = m > 0, \text{ then either } \exists\, i \varepsilon C_1(\underline{X}) \text{ s.t. } L_\pi(0_i, \underline{X}) = m-1,\ i = \pi(0_i, \underline{X}), \text{ or } \exists\, i \varepsilon C_0(\underline{X}) \text{ s.t. } L_\pi(1_i, \underline{X}) = m \quad . \tag{4.1.5}$$

$$\text{If } L_\pi(\underline{X}) = m > 0, \text{ then } \exists\, i \varepsilon C_1(\underline{X}) \text{ s.t. } L_\pi(0_i, \underline{X}) \leq m-1. \tag{4.1.6}$$

Proof: (4.1.2) Any state may be reached by repairing all components and then allowing components in $C_0(\underline{X})$ to fail.

(4.1.3) In order to reach $\underline{X}$ from $\underline{0}$, every component in $C_1(\underline{X})$ must be repaired at least once.

(4.1.4) One way to reach $\underline{Y} \leq \underline{X}$ is to reach $\underline{X}$ first and then allow the components in $C_1(\underline{X}) - C_1(\underline{Y})$ to fail.

(4.1.5) Let $\underline{X}$ be such that $L_\pi(\underline{X}) = m$, and let $\underline{0}, \underline{X}_1, \underline{X}_2, \ldots, \underline{X}_k, \underline{X}$ be the successive states passed through in reaching $\underline{X}$ from $\underline{0}$ using m repairs. Each state $\underline{X}_i$ is obtained from the previous state $\underline{X}_{i-1}$ by one repair or one failure. At least one such path is possible by the definition of $L_\pi(\underline{X})$.

We have two possibilities. Either the transition from $\underline{X}_k$ to $\underline{X}$ is from a repair, or the transition from $\underline{X}_k$ to $\underline{X}$ is from a failure.

In the first case $L_\pi(\underline{X}_k) \leq m-1$, since we have a path from $\underline{0}$ to $\underline{X}_k$ with m-1 repairs. In addition, $L_\pi(\underline{X}_k) \geq$ m-1 since the existence of a path to $\underline{X}_k$ with fewer than m-1 repairs implies the existence of a path to $\underline{X}$ with fewer than m repairs, a contradiction. Thus $L_\pi(\underline{X}_k) = m-1$. Choose $i = C_1(\underline{X}) - C_1(\underline{X}_k)$ to satisfy the first condition on the right hand side of (4.1.5).

If the transition from $\underline{X}_k$ to $\underline{X}$ is from a failure, we conclude in an analogous matter than $L_\pi(\underline{X}_k) = m$. Now choose $i = C_1(\underline{X}_k) - C_1(\underline{X})$ to satisfy the second condition on the right hand side of (4.1.5).

(4.1.6) Again let $\underline{0}, \underline{X}_1, \ldots, \underline{X}_k, \underline{X}$ be the successive

states passed through in reaching $\underline{X}$ from $\underline{0}$ using m repairs. Let the last repair take place in state $\underline{X}_e$. Then by the argument used in the proof of (4.1.5), $L_\pi(\underline{X}_e) = m-1$.

Let $i = \pi(\underline{X}_e)$, then $i \varepsilon C_1(\underline{X})$, since if not by (4.1.4) $\underline{X} \leq \underline{X}_e \Rightarrow L_\pi(\underline{X}) \leq m-1$, a contradiction.

However $0_i, \underline{X} \leq \underline{X}_e$ since the last repair (which was component i) took place at $\underline{X}_e$. Thus (4.1.4) tells us $L_\pi(0_i, \underline{X}) \leq m-1$ and (4.1.6) is proven.

End of proof of Theorem 4.1 A.

Theorem 4.1 B. Let the failure rate of the $i^{\underline{th}}$ component be μ_i, and the repair rate of the $i^{\underline{th}}$ component be $k\lambda_i$. Let π be a candidate policy and $e_\pi(\underline{X})$ be the ergodic probability of state $\underline{X}$ under policy π (note that the dependence on k has been suppressed). Then

$$\lim_{k \to 0} (k^{1-L_\pi(\underline{X})}) e_\pi(\underline{X}) = 0 \quad . \tag{4.1.7}$$

Proof: The proof is by induction on $L_\pi(\underline{X})$. Clearly (4.1.7) holds when $L_\pi(\underline{X}) = 0$ since $e_\pi(\underline{X})$ is bounded. Now assume that (4.1.7) is true for $\underline{X}$ such that $L_\pi(\underline{X}) = m < n$.

Equating the ergodic rate of entering and leaving the group of states $G_m = \{\underline{X} | L_\pi(\underline{X}) \leq m\}$ yields the following equation.

$$\sum_{\underline{X}:L_\pi(\underline{X})>m} e_\pi(\underline{X}) \quad \sum_{\substack{i\varepsilon C_1(\underline{X}) \\ L_\pi(0_i,\underline{X})\leq m}} \mu_i \quad = \tag{4.1.8}$$

$$\sum_{\substack{\underline{X}:L_\pi(\underline{X})=m \\ L_\pi(1_{\pi(\underline{X})},\underline{X})>m}} k\lambda_{\pi(\underline{X})} e_\pi(\underline{X})$$

We note that the restrictions on the summations in (4.1.8) are correct as follows. (4.1.4) tells us that G_m can be entered only through failures, and left only through repairs. Also $L_\pi(\underline{X}) \leq m$, $L_\pi(1_{\pi(\underline{X})},\underline{X}) > m \Rightarrow L_\pi(\underline{X}) = m$.

Now multiply both sides of (4.1.8) by k^{-m}. By the induction hypothesis the limit of the right hand side as k goes to zero is zero. Since all terms on the left hand side are positive, they must all have zero limit. (4.1.6) tells us that every $\underline{X}$ such that $L_\pi(\underline{X}) = m+1$ has a term on the left hand side which is a non-zero constant coefficient multiplied by $k^{-m}e_\pi(\underline{X})$. We have thus proven (4.1.7) by induction. End of proof of Theorem 4.1 B.

Theorem 4.1 C. Under the conditions of Theorem 4.1 B

$$\lim_{k\to 0} k^{-L_\pi(\underline{X})} e_\pi(\underline{X}) = f_\pi(\underline{X}) \quad , \tag{4.1.9}$$

where $0 < f_\pi(\underline{X}) < \infty$. Furthermore, $f_\pi(\underline{0}) = 1$ and for $\underline{X} \neq \underline{0}$

$$\Big(\sum_{i\in C_1(\underline{X})} \mu_i\Big) f_\pi(\underline{X}) = \sum_{\substack{i\in C_1(\underline{X})\\ i=\pi(0_i,\underline{X})\\ L_\pi(0_i,\underline{X})=L_\pi(\underline{X})-1}} \lambda_i f_\pi(0_i,\underline{X}) + \sum_{\substack{i\in C_0(\underline{X})\\ L_\pi(1_i,\underline{X})=L_\pi(\underline{X})}} \mu_i f_\pi(1_i,\underline{X}) \; . \tag{4.1.9}$$

Equation (4.1.9) allows computation of $f_\pi(\underline{X})$ recursively starting with the lowest values of $L_\pi(\underline{X})$ and maximal elements of sets of constant $L_\pi(\underline{X})$.

<u>Proof</u>: The proof again is by induction on L_π. (4.1.7) tells us that $\lim_{k\to 0} e_\pi(\underline{X}) = 0$, $\underline{X} \neq \underline{0}$, thus $\lim_{k\to 0} e_\pi(\underline{0}) = 1$, establishing the theorem when $L_\pi = 0$.

Now assume the theorem is true for all $\underline{X}$ s.t. $L_\pi(\underline{X}) \leq m < n$. Write the balance equation for $e_\pi(\underline{X})$ as follows:

$$\Big(\sum_{i\in C_1(\underline{X})} \mu_i + k\lambda_{\pi(\underline{X})}\Big) e_\pi(\underline{X}) = \sum_{\substack{i\in C_1(\underline{X})\\ i=\pi(0_i,\underline{X})}} k\lambda_i e_\pi(0_i,\underline{X}) + \sum_{i\in C_0(\underline{X})} \mu_i e_\pi(1_i,\underline{X}) \; . \tag{4.1.10}$$

Assume $L_\pi(\underline{X}) = m+1$. Multiply both sides of (4.1.10) by $k^{-(m+1)}$. By Theorem 4.1 B, the second term on the left hand side has zero limit as k goes to zero. Now

consider the first sum on the right hand side. $i = \pi(0_i, \underline{X}) \Rightarrow L_\pi(0_i, \underline{X}) \geq m$, so that by Theorem 4.1 B a term in this sum will have xero limit unless $L_\pi(0_i, \underline{X}) = m$. In this case we know the limit by the induction hypothesis. Now consider the second sum on the right hand side. Since $L_\pi(1_i, \underline{X}) \geq L_\pi(\underline{X})$ by (4.1.4), $L_\pi(1_i, \underline{X}) \geq m+1$. Unless $L_\pi(1_i, \underline{X}) = m+1$ a term in this sum will have zero limit by Theorem 4.1 B.

Now let $\underline{X}_0$ be a maximal element of $\{\underline{X} | L_\pi(\underline{X}) = m+1\}$. A maximal element of sets of binary n vectors always exists. By the reasoning of the previous paragraph the theorem is true for $\underline{X}_0$. Non-zero of $f_\pi(\underline{X}_0)$ follows from (4.1.5).

Let $\underline{X}_1$ be a maximal element of $\{\underline{X} | L_\pi(\underline{X}) = m+1\} - \{\underline{X}_0\}$. By the same reasoning the theorem is true for $\underline{X}_1$, and eventually for every element of $\{X | L_\pi(\underline{X}) = m+1\}$. End of induction, and end of proof of Theorem 4.1 C.

Note that higher order approximations similar to those of Theorem 3.1 D may be obtained in a similar manner if desired. The statement and proof are omitted.

4.2 Limiting Optimal Ergodic Reliability of the System

Theorem 4.2 A. Let R(k) be the optimal reliability of the previously described system. Let m_0 = min cardinality of the path sets = min cardinality of the min path sets. Then

$$\lim_{k\to 0} k^{-m_0} R(k) = C_0 \qquad , \tag{4.2.1}$$

where

$$C_0 = \max_{\substack{(i_1,\ldots,i_{m_0})\\ \text{is a path set}}} \left[\frac{\lambda_{i_1} \quad \lambda_{i_2} \quad \cdots \quad \lambda_{i_{m_0}}}{(\mu_{i_1})(\mu_{i_1}+\mu_{i_2})\cdots(\mu_{i_1}+\cdots+\mu_{i_{m_0}})}\right] , \tag{4.2.2}$$

without loss of generality the elements $i_1, i_2, \ldots, i_{m_0}$ may be arranged so that $\mu_{i_1} \le \mu_{i_2} \le \cdots \le \mu_{i_{m_0}}$.

Furthermore, for all $k > k_0$ for some $k_0 > 0$ any candidate policy π' which does not repair i_{n+1} in the state $1_{i_1}, 1_{i_2}, \ldots, 1_{i_n}, \underline{0}$ $\forall$ $n \le m_0-1$ for some set $\{i_1,\ldots,i_{m_0}\}$ which achieves the maximum in (4.2.2) is not optimal.

Proof: Let π^0 be a policy which repairs i_{n+1} in the state $1_{i_1}, 1_{i_2}, \ldots, 1_{i_n}$, $\underline{0}$ $\forall$ $n \le m_0-1$, for a set $\{i_1,\ldots,i_{m_0}\}$ which maximizes (4.2.2).

Then π^0 has no working states $\underline{X}$ with $L_{\pi^0}(\underline{X}) < m_0$ by (4.1.3), and exactly one working state $\underline{X}$ with $L_{\pi^0}(\underline{X}) = m_0$,

the state $1_{i_1},\ldots,1_{i_{m_0}},\underline{0}$, since any other states $\underline{X}$ with $L_{\pi^0}(\underline{X}) = m_0$ must have $C_1(\underline{X}) < m_0$.

Let $R^0(k)$ be the ergodic reliability of the system under policy π^0. By Theorem 4.1 C and the last paragraph together with the fact that the number of states is finite, we obtain

$$\lim_{k\to 0} k^{-m_0}R^0(k) = f_{\pi^0}(1_{i_1},\ldots,1_{i_{m_0}},\underline{0}) \quad . \tag{4.2.3}$$

But (4.1.9) yields

$$f_{\pi^0}(1_{i_1},\ldots,1_{i_{m_0}},\underline{0}) = C_0 \quad , \tag{4.2.4}$$

where C_0 is given by (4.2.2).

The policy π' can have no working states $\underline{X}$ with $L_{\pi'}(\underline{X}) < m_0$, and at most one working state $\underline{X}$ with $L_{\pi'}(\underline{X}) = m_0$. This will be the case if it mimics policy π^0, but for a path set $\{i_1,\ldots,i_{m_0}\}$ which does not maximize the expression in (4.2.2). Regardless, we conclude from Theorem 4.1 C that if $R'(k)$ is the ergodic reliability of policy π',

$$\lim_{k\to 0} k^{-m_0}R'(k) < C_0 \quad . \tag{4.2.5}$$

Comparison of (4.2.5) and (4.2.3) shows that π' is not optimal for $k < k_0$.

The theorem follows from the fact that the number of candidate policies is finite. End of proof of Theorem 4.2 A.

4.3 Limiting Expected Passage Times

Some of the terminology in section 4.3 will be identical with terminology in section 3.3 although having a different meaning. Chapters 3 and 4 are independent and no confusion should arise for the reader.

Theorem 4.3 A. For the system described in Theorem 4.1 B, let $T_\pi(k,\underline{X})$ be the expected time to go from state $\underline{X}$ to state $\underline{0}$ under candidate policy π. Then

$$\lim_{k\to 0} T_\pi(k,\underline{X}) = T(0,\underline{X}) < \infty \quad . \tag{4.3.1}$$

Proof: The equations for the expected passage times yield solutions continuous in k. However, the passage time to $\underline{0}$ in the absence of repairs is finite and independent of policy. End of proof of Theorem 4.3 A.

Theorem 4.3 B. Let $G_\pi(k,\underline{X})$ be the expected time the system is working during passage from $\underline{X}$ to $\underline{0}$ under candidate policy π. $G_\pi(k,\underline{0}) = 0$. Let $K_\pi(\underline{X})$ be the minimum number of repairs under policy π (allowing any number of failures) to reach a working state from $\underline{X}$ without passing through $\underline{0}$. We make the convention that $\phi(\underline{X}) = 1 \Rightarrow K_\pi(\underline{X}) = 0$. Then

$$\lim_{k\to 0} k^{1-K_\pi(\underline{X})} G_\pi(k,\underline{X}) = 0 \quad . \tag{4.3.2}$$

<u>Proof</u>: The proof is by induction on $K_\pi(\underline{X})$. Theorem 4.3 A establishes (4.3.2) when $K_\pi(\underline{X}) = 0$.

Now assume (4.3.2) is true for $\underline{X}$ such that $K_\pi(\underline{X}) \leq m$ and assume that if $K_\pi(\underline{X}) \geq m$, then

$$\lim_{k \to 0} k^{1-m} G_\pi(k,\underline{X}) = 0 \quad . \tag{4.3.3}$$

The following identity is easily obtained in the standard manner. If $\underline{X} \neq \underline{0}$,

$$\Big(\sum_{i \varepsilon C_1(\underline{X})} \mu_i + k\lambda_{\pi(\underline{X})}\Big) G_\pi(k,\underline{X}) = \delta_{1\phi(\underline{X})} + \sum_{i \varepsilon C_1(\underline{X})} \mu_i G_\pi(k,0_i,\underline{X}) + k\lambda_{\pi(\underline{X})} G_\pi(k,1_{\pi(\underline{X})},\underline{X}) \tag{4.3.4}$$

Now suppose $K_\pi(\underline{X}) \geq m+1$ which implies $\phi(\underline{X}) = 0$. Multiply both sides of (4.3.4) by k^{-m}. The induction hypothesis (4.3.3) tells us the second term on the left hand side has limit zero as k goes to zero. We note in general $K_\pi(1_{\pi(\underline{X})},\underline{X}) \geq K_\pi(\underline{X})-1$, since one possible way to reach a given working state from $\underline{X}$ is to go through $(1_{\pi(\underline{X})},\underline{X})$. Specifically $K_\pi(1_{\pi(\underline{X})},\underline{X}) \geq m$, and the induction hypothesis (4.3.3) tells us the last term on the right hand side has limit zero.

We note that $K_\pi(0_i,\underline{X}) \geq K_\pi(\underline{X})$, so that (4.3.4) and the arguments of the previous chapter show that if $\underline{X}_0$ is a

minimal element of $\{\underline{X} \mid K_\pi(\underline{X}) \geq m+1\} - \{\underline{0}\}$ then

$$\lim_{k \to 0} k^{-m} G_\pi(k, \underline{X}_0) = 0.$$

Similarly we establish the same result for a minimal element of $\{\underline{X} \mid K_\pi(\underline{X}) \geq m+1\} - \{\underline{0}\} - \{\underline{X}_0\}$ and so forth for all $\underline{X}$ with $K_\pi(\underline{X}) \geq m+1$. End of the induction proof and end of the proof of Theorem 4.3 B.

<u>Theorem 4.3 C</u>.

$$\lim_{k \to 0} k^{-K_\pi(\underline{X})} G_\pi(k, \underline{X}) = C_\pi(\underline{X}) \tag{4.3.5}$$

where $0 < C_\pi(\underline{X}) < \infty$, $\underline{X} \neq \underline{0}$.

For $\underline{X}$ such that $\phi(\underline{X}) = 0$,

$$\Big(\sum_{i \in C_1(\underline{X})} \mu_i\Big) C_\pi(\underline{X}) = \sum_{\substack{i \in C_1(\underline{X}) \\ K_\pi(0_i, \underline{X}) = K_\pi(\underline{X})}} \mu_i C_\pi(0_i, \underline{X}) + \tag{4.3.6}$$

$$\delta_{K_\pi(\underline{X}), K_\pi(1_{\pi(\underline{X})}, \underline{X})+1} \; \lambda_{\pi(\underline{X})} C_\pi(1_{\pi(\underline{X})}, \underline{X}) \quad .$$

<u>Proof</u>: The theorem is proven for $K_\pi(\underline{X}) = 0$ by a continuity argument similar to that used in Theorem 4.3 A. The rest of the proof is based on (4.3.4) and is similar to previous proofs of this type given and is omitted. End of proof of Theorem 4.3 C.

4.4 Limiting Optimal State Actions Under First Order Passage Times

Theorem 4.4 A. Suppose that for two candidate policies π^0 and π'

$$\frac{G_{\pi^0}(k,\underline{X})}{T_{\pi^0}(k,\underline{X})} \geq b \quad \text{(ergodic probability the system works under } \pi'\text{)} \tag{4.4.1}$$

for $k < k_0$ with $k_0 > 0$, $b > 0$, and

$$G_{\pi^0}(k,\underline{X}) > yG_{\pi'}(k,\underline{X}) \tag{4.4.2}$$

for $k < k_0$ and $y > 1$.

Then as k goes to zero, π' is not optimal.

Proof: We contradict the optimality of π' as follows. Construct a policy π'' which uses policy π^0 between $\underline{X}$ and $\underline{1}$ and uses π' between $\underline{1}$ and $\underline{X}$.

Let $S(k,\underline{X})$ be the expected time to pass from $\underline{1}$ to $\underline{X}$ under policy π', and let $F(k,\underline{X})$ be the expected time the system works during passage from $\underline{1}$ to $\underline{X}$ under policy π'. By alternating renewal theory,[36,37] the ergodic probability that the system is working under π' is

$$P_{\pi'}(\phi(\underline{X})=1) = \frac{G_{\pi'}(k,\underline{X}) + F(k,\underline{X})}{T_{\pi'}(k,\underline{X}) + S(k,\underline{X})} \tag{4.4.3}$$

and the ergodic probability that the system works under policy π'' is

$$P_{\pi''}(\phi(\underline{X}) = 1) = \frac{G_{\pi^0}(k,\underline{X}) + F(k,\underline{X})}{T_{\pi^0}(k,\underline{X}) + S(k,\underline{X})} . \tag{4.4.4}$$

Now suppose that π'' is not strictly better than π', or that

$$P_{\pi''}(\phi(\underline{X}) = 1) \leq P_{\pi'}(\phi(\underline{X}) = 1) . \tag{4.4.5}$$

Substitution from (4.4.3) and (4.4.4) into (4.4.5) and suppression of the arguments gives

$$T_{\pi 0} \geq T_{\pi'} + \frac{(T_{\pi'}+S)}{(G_{\pi'}+F)} (G_{\pi 0} - G_{\pi'}) \tag{4.4.6}$$

Using (4.4.2) gives

$$T_{\pi 0} \geq T_{\pi'} + (1-\frac{1}{y}) G_{\pi 0} \frac{(T_{\pi'}+S)}{(G_{\pi'}+F)} \tag{4.4.7}$$

for $k < k_0$.

Now, using (4.4.1) and (4.4.3), we obtain

$$T_{\pi 0} \geq T_{\pi'} + b(1-\frac{1}{y}) T_{\pi 0} \tag{4.4.8}$$

for $k < k_0$, a contradiction as $k \to 0$ since

$$\lim_{k \to 0} T_{\pi 0} = \lim_{k \to 0} T_{\pi'} \quad ,$$

by Theorem 4.3 A. End of proof of Theorem 4.4 A.

<u>Theorem 4.4 B</u>. A candidate policy π' such that $K_{\pi'}(\underline{X}) \neq \min_{\pi} K_{\pi}(\underline{X})$ is not optimal for $k < k'$ for some $k' > 0$. Furthermore, a candidate policy π' such that

$$C_{\pi'}(\underline{X}) \neq \max_{\pi 0 : K_{\pi 0}(\underline{X}) = \min_{\pi} K_{\pi}(\underline{X})} [C_{\pi 0}(\underline{X})]$$

is not optimal for $k < k'$ for some $k' > 0$.

Proof: Compare π' with π^0 where π^0 has both of the above properties. By (4.3.5) we have that

$$\lim_{k \to 0} k^{-K_{\pi^0}(\underline{X})} G_{\pi^0}(\underline{X}) = C_{\pi^0}(\underline{X}) \tag{4.4.9}$$

But $K_{\pi^0} \leq m_0$ = min cardinality of path sets. Therefore, (4.4.9), (4.2.1), and (4.3.1) imply condition (4.4.1) of Theorem 4.4 A.

Also Theorem 4.3 C implies condition (4.4.2) of Theorem 4.4 A, and the remainder of the theorem follows from 4.4 A. End of proof of Theorem 4.4 B.

Theorem 4.4 B and (4.3.6) give us a valuable technique to solve for the limiting optimal policy.

DIVISION III

Specific Cases

In the next two chapters we treat various examples of specific structure functions and or specific repair or failure rates.

The two component parallel system shows that, in general, such specific cases are extremely complicated. This system has a symmetric structure function of only two components, and there are only two candidate policies to consider. Nevertheless, the region of optimality for each candidate policy is somewhat complicated.

Let π_1 be the candidate policy which repairs component 1 when both components are failed, and let π_2 be the candidate policy which repairs component 2 when both components are failed.

If P_{ij} is the ergodic probability of (i,j) under policy 1, then by standard methods P_{ij} can be computed from the following equations:

$$\begin{aligned}
(\mu_1+\mu_2)P_{11} &= \lambda_2 P_{10}+\lambda_1 P_{01} \\
(\lambda_2+\mu_1)P_{10} &= \mu_2 P_{11}+\lambda_1 P_{00} \\
(\lambda_1+\mu_2)P_{01} &= \mu_1 P_{11} \\
\lambda_1 P_{00} &= \mu_1 P_{10}+\mu_2 P_{01}
\end{aligned} \qquad (5.0.1)$$

$$P_{00} + P_{01} + P_{01} + P_{11} = 1 \quad .$$

The solution to the previous equations is:

$$\begin{aligned} P_{00} &= d^{-1}\mu_1\mu_2(\mu_1+\mu_2+\lambda_1+\lambda_2) \\ P_{01} &= d^{-1}\mu_1\lambda_1\lambda_2 \\ P_{10} &= d^{-1}\mu_2\lambda_1(\mu_1+\mu_2+\lambda_1) \\ P_{11} &= d^{-1}\lambda_1\lambda_2(\mu_2+\lambda_1) \quad , \end{aligned} \tag{5.0.2}$$

where

$$d = (\mu_1+\lambda_1)(\mu_2\lambda_1+\mu_2\lambda_2+\mu_1\mu_2+\mu_2^{\;2}+\lambda_1\lambda_2) \quad .$$

The ergodic probabilities under policy π_2 are obtained by symmetry by exchange of the subscripts in the last equations. π_1 is easily seen to be optimal if

$$\begin{aligned} &\mu_1\mu_2\lambda_1+\mu_2\lambda_1^{\;2}+\mu_2^{\;2}\lambda_1+\lambda_1^{\;2}\lambda_2 \quad \geq \\ &\mu_1\mu_2\lambda_2+\mu_1\lambda_2^{\;2}+\mu_1^{\;2}\lambda_2+\lambda_2^{\;2}\lambda_1 \end{aligned} \tag{5.0.3}$$

a criterion not expressible in the form: $f(\mu_1,\lambda_1) \geq g(\mu_2,\lambda_2)$.

The results from Chapters 3 and 4 tell us that for highly reliable systems we should repair the component with largest λ_i first, for highly unreliable systems we should repair the component with

largest ratio $\frac{\lambda_i}{\mu_i}$.

One result we can easily prove is the following: if a) $\lambda_1 \geq \lambda_2$ and b) $\frac{\lambda_1}{\mu_1} \geq \frac{\lambda_2}{\mu_2}$, then for the two component parallel system it is optimal to repair the first component first.

Proof:

$$\mu_1\mu_2[(\lambda_1+\lambda_1(\frac{\lambda_1}{\mu_1})+\mu_2(\frac{\lambda_1}{\mu_1})+\lambda_1(\frac{\lambda_1}{\mu_1})(\frac{\lambda_2}{\mu_2}))] \geq$$

$$\mu_1\mu_2[\mu_1(\frac{\lambda_2}{\mu_2})+\lambda_1(\frac{\lambda_2}{\mu_2})+\lambda_2+\lambda_1(\frac{\lambda_1}{\mu_1})(\frac{\lambda_2}{\mu_2})] \quad ,$$

by inequality b; and is $\geq$

$$\mu_1\mu_2[\mu_1(\frac{\lambda_2}{\mu_2})+\lambda_2(\frac{\lambda_2}{\mu_2})+\lambda_2+\lambda_2(\frac{\lambda_1}{\mu_1})(\frac{\lambda_2}{\mu_2})] \quad ,$$

by inequality a,

and thus π_1 is optimal by (5.0.3).

Whether or not the fact that the same action is optimal for both highly reliable and highly unreliable systems implies it is also optimal for all values of the parameter k given in chapters 3 and 4 is an open question.

CHAPTER 5

The Series System

The structure function for the series system is $\phi(\underline{X}) = \Pi X_i$. We note that there is only one state vector $\underline{X}$ for which $\phi(\underline{X}) = 1$, this is $\underline{X} = \underline{1}$. Since the dwell time in the working state and transitions to other states are independent of policy, alternating renewal theory tells us that maximizing the ergodic probability of the working state is equivalent to minimizing the expected time until the working state is reached.

5.1 Series System with $\mu_i \equiv \mu$

We first consider the case where the failure rates are identical, $\mu_i = \mu$, for every i. We arrive at the surprising conclusion that the policy doesn't matter! The heuristic reasoning that all components must be repaired anyway, so that the policy obviously doesn't matter; is spurious because components are subjected to failure during the course of repair of other components, and will be contradicted when the failure rates are not identical.

Theorem 5.1 A. Let π be a candidate policy (see section 1.4), and let $T_\pi(\underline{X})$ be the random variable which is the time to reach state $\underline{1}$ from state $\underline{X}$ under policy π. Then

$$E[e^{-sT_\pi(\underline{X})}] = \frac{L(\underline{X})}{L(\underline{1})} \tag{5.1.1}$$

where

$$L(\underline{X}) = 1 + s \sum_{i \in C_1(\underline{X})} \frac{1}{\lambda_i} + s(s+\mu) \sum_{\substack{i<j \\ i,j \in C_1(\underline{X})}} \frac{1}{\lambda_i \lambda_j} + \dots + s(s+\mu)\dots(s+(|C_1(\underline{X})|-1)\mu) \prod_{i \in C_1(\underline{X})} \frac{1}{\lambda_i} \tag{5.1.2}$$

Proof: Let $L_\pi(\underline{X}) = E[e^{-sT_\pi(\underline{X})}]$.

We obtain the following set of equations for $\underline{X} \neq 1$ by standard techniques.[24]

$$(\lambda_{\pi(\underline{X})} + |C_1(\underline{X})|\mu + s)L_\pi(\underline{X}) =$$

$$\lambda_{\pi(\underline{X})} L_\pi(1_{\pi(\underline{X})},\underline{X}) + \mu \sum_{j\varepsilon C_1(\underline{X})} L_\pi(0_j,\underline{X}) \quad . \tag{5.1.3}$$

These equations, with the boundary condition $L_\pi(\underline{1}) = 1$ uniquely determine $L_\pi(\underline{X})$. We show that $L(\underline{X})$ given by (5.1.2) satisfies (5.1.3), thus establishing (5.1.1).

It is readily seen from (5.1.2) that for $\underline{X} \neq \underline{1}$,

$$\lambda_{\pi(\underline{X})}[L(1_{\pi(\underline{X})},\underline{X}) - L(\underline{X})] =$$

$$s + s(s+\mu) \sum_{i\varepsilon C_1(\underline{X})} \frac{1}{\lambda_i} + \dots + \tag{5.1.4}$$

$$s(s+\mu)\dots(s+|C_1(\underline{X})|\mu) \prod_{i\varepsilon C_1(\underline{X})} \frac{1}{\lambda_i} \quad .$$

It also follows from (5.1.2) that

$$\sum_{j\varepsilon C_1(\underline{X})} L(0_j,\underline{X}) = |C_1(\underline{X})| +$$

$$(|C_1(\underline{X})|-1)s \sum_{i\varepsilon C_1(\underline{X})} \frac{1}{\lambda_i} + (|C_1(\underline{X})|-2)s(s+\mu) \sum_{\substack{i<j \\ i,\, j\varepsilon C_1(\underline{X})}} \frac{1}{\lambda_i\lambda_j} +$$

$$\dots + s(s+\mu)\dots(s+(|C_1(\underline{X})|-2)\mu) \frac{\sum_{i\varepsilon C_1(\underline{X})} \lambda_i}{\prod_{j\varepsilon C_1(\underline{X})} \lambda_j} \quad , \tag{5.1.5}$$

and (5.1.2), (5.1.5), and (5.1.4) check when substituted in

(5.1.3) since the factors which multiply $\Sigma\frac{1}{\lambda_{i_1}\cdots\lambda_{i_m}}$ are found to be equal by the following readily verified identity.

$$(|C_1(\underline{X})|\mu+s)[s(s+\mu)\dots(s+(m-1)\mu)] = s(s+\mu)\dots(s+m\mu) + \mu[|C_1(\underline{X})|-m]s(s+\mu)\dots(s+(m-1)\mu) \quad . \tag{5.1.6}$$

End of proof of Theorem 5.1 A.

We note that the previous theorem is valid also for simultaneous effort or mixed policies, provided full use is made of the repairman. The proof is similar.

Note that Theorem 5.1 A says that the time to repair the system is stochastically independent of the candidate policy, and thus the expected time to repair the system is independent of candidate policy. We conclude, by alternating renewal theory that the ergodic probability that the system works is independent of candidate policy.

We also easily obtain by standard methods the expected time to fix the system:

$$E[T_\pi(\underline{X})] = \frac{-d}{ds}L_\pi(\underline{X})\Big|_{s=0} = \frac{-L'(\underline{X})L(\underline{1})+L(\underline{X})L'(\underline{1})}{(L(\underline{1}))^2}\Bigg|_{s=0} \tag{5.1.7}$$

or that

$$E[T_\pi(\underline{X})] = \sum_{\{i\}\not\subset C_1(\underline{X})} \frac{1}{\lambda_i} + \sum_{\substack{i<j \\ \{i,j\}\not\subset C_1(\underline{X})}} \frac{\mu}{\lambda_i\lambda_j} + \qquad (5.1.8)$$

$$2! \sum_{\substack{i<j<k \\ \{i,j,k\}\not\subset C_1(\underline{X})}} \frac{\mu^2}{\lambda_i\lambda_j\lambda_k} + \ldots + \frac{(n-1)!\mu^{n-1}}{\prod_i \lambda_i}$$

The ergodic probability that the system works under any candidate policy is readily available through alternating renewal theory, or from the ergodic probability of the time reversible policy whose potentials are given in (2.2.2). Let $e(\underline{1})$ be the ergodic probability of $\underline{1}$ under any candidate policy. Then

$$e(\underline{1}) = \frac{1}{1+\sum_i \frac{\mu}{\lambda_i} + 2! \sum_{i<j} \frac{\mu^2}{\lambda_i\lambda_j} + \ldots + n! \prod_{i=1}^{n} (\frac{\mu}{\lambda_i})} \qquad (5.1.9)$$

We can also easily compute

$$F(\pi,s,\underline{1}) = E_\pi(\int_0^\infty e^{-st}\phi(\underline{X}(t))dt \,|\, \underline{X}(0)=\underline{1})dt \quad ,$$

the expected integral of the discounted time that the system works, by an alternating renewal theory approach. The expected value of this integral for the first working period is

$$\int_0^\infty n\mu e^{-n\mu t}[\frac{1}{s}(1-e^{-st})]dt = \frac{1}{n\mu+s} \quad .$$

The expected discounted time until the system begins its second working period is $E(e^{-sT_{\underline{1}\underline{1}}}) =$

$$\frac{n}{n\mu+s}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{L(0_i,\underline{1})}{L(\underline{1})}\right] = \frac{\mu}{(n\mu+s)L(\underline{1})}\left[n+(n-1)s\sum_i \frac{1}{\lambda_i} + (n-2)s(s+\mu)\sum_{i<j}\frac{1}{\lambda_i\lambda_j} + \dots + s(s+\mu)\dots(s+(n-2)\mu)\frac{\sum_i \lambda_i}{\prod_j \lambda_j}\right]$$

We thus obtain by alternating renewal theory

$$F(\pi,s,\underline{1}) = \frac{1}{n\mu+s} + E(e^{-sT_{\underline{1}\underline{1}}})F(\pi,s,\underline{1}) \qquad (5.1.10)$$

We conclude that:

$$F(\pi,s,\underline{1}) = \qquad (5.1.11)$$

$$\frac{L(\underline{1})}{s+s(s+\mu)\sum \frac{1}{\lambda_i} + s(s+\mu)(s+2\mu)\sum_{i<j}\frac{1}{\lambda_i\lambda_j} + \dots + s(s+\mu)\dots(s+n\mu)\frac{1}{\prod_i \lambda_i}}$$

and that

$$F(\pi,s,\underline{X}) = \frac{L(\underline{X})}{L(\underline{1})}F(\pi,s,\underline{1}) \quad . \qquad (5.1.12)$$

5.2 The Two Component Series System

We now relax the assumption that the failure rates of all components are identical. We start with a two component series system.

Again, there are only two candidate policies to consider. Let π_1 be the candidate policy which repairs component 1 when both components are failed, and define π_2 similarly. If $t_i(\underline{X})$ denotes the expected time to go from $\underline{X}$ to $\underline{1}$ under policy π_i, then we obtain easily:

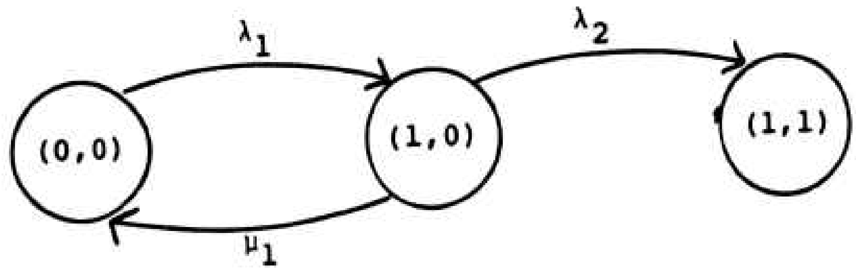
$$\begin{aligned} t_1((0,0)) &= \frac{1}{\lambda_1} + t_1((1,0)) \\ t_1((1,0)) &= \frac{1}{\lambda_2+\mu_1} + \frac{\mu_1}{\lambda_2+\mu_1}\, t_1((0,0)) \end{aligned} \tag{5.2.1}$$

which has the solution

$$\begin{aligned} t_1((0,0)) &= \frac{\lambda_1+\lambda_2+\mu_1}{\lambda_1\lambda_2} \\ t_2((0,0)) &= \frac{\lambda_1+\lambda_2+\mu_2}{\lambda_1\lambda_2} \quad , \end{aligned} \tag{5.2.2}$$

where t_2 is obtained by interchanging the subscripts in the expression for t_1 by symmetry. Thus π_1 is superior to π_2 if $\mu_1 < \mu_2$, a relatively simple criterion. It is somewhat surprising that the repair rates are irrelevant for determination of the optimal policy.

We are now interested in the distribution of time to reach (1,1) from (0,0) under policy π_1. We proceed in the following standard manner. Make the state (1,1) an absorbing state. The system evolution is then governed by a continuous time Markov chain with the following transition diagram.[24,37]



The negatives of the eigenvalues[24] are then r_1 and r_2 where

$$\begin{aligned} r_1 &= 1/2[(\lambda_1+\lambda_2+\mu_1) + \sqrt{(\lambda_1+\lambda_2+\mu_1)^2-4\lambda_1\lambda_2}] \\ r_2 &= 1/2[(\lambda_1+\lambda_2+\mu_1) - \sqrt{(\lambda_1+\lambda_2+\mu_1)^2-4\lambda_1\lambda_2}] \end{aligned} \tag{5.2.3}$$

The system evolution conditioned on $\underline{X}(0) = \underline{0}$ is

$$\begin{pmatrix} P\{\underline{X}(t) = (0,0)\} \\ \\ P\{\underline{X}(t) = (1,0)\} \end{pmatrix} = \begin{pmatrix} \dfrac{\lambda_1-r_2}{r_1-r_2}\, e^{-r_1 t} + \dfrac{r_1-\lambda_1}{r_1-r_2}\, e^{-r_2 t} \\ \\ \dfrac{-\lambda_1}{r_1-r_2}\, e^{-r_1 t} + \dfrac{\lambda_1}{r_1-r_2}\, e^{-r_2 t} \end{pmatrix}$$

(5.2.4)

Thus if X_{i1} is the random variable denoting the passage time from (0,0) to (1,1) under policy π_i, then

$$\overline{F}_{11}(t) = \frac{-r_2}{r_1-r_2} e^{-r_1 t} + \frac{r_1}{r_1-r_2} e^{-r_2 t} \qquad (5.2.5)$$

where $\overline{F}_{11}(t) = P\{X_{11} > t\}$.

Note that $\overline{F}_{21}(t)$ is also determined by (5.2.5) where r_1 and r_2 are determined from (5.2.3) by replacing μ_1 with μ_2. Thus, regarding $\overline{F}_{11}(t)$ and r_1 and r_2 as functions of μ_1,

$$\overline{F}_{21}(t) = \overline{F}_{11}(t) + \int_{\mu_1}^{\mu_2} \frac{\partial \overline{F}_{11}(t)}{\partial \mu_1} d\mu_1 \quad . \qquad (5.2.6)$$

<u>Theorem 5.2 A</u>. If $\mu_1 \le \mu_2$ then $X_{11} \underset{st}{\le} X_{21}$, or equivalently $\overline{F}_{11}(t) \le \overline{F}_{21}(t)\ \forall\ t \ge 0$. Furthermore, if $\mu_1 < \mu_2$, then $\overline{F}_{11}(t) < \overline{F}_{21}(t)\ \forall\ t > 0$. If $\mu_1 = \mu_2$, $X_{11} \underset{st}{=} X_{21}$.

<u>Proof</u>: We use (5.2.6) and show that $\frac{\partial \overline{F}_{11}(t)}{\partial \mu_1} > 0$ for $t > 0$.

We first note that from (5.2.3) $r_1 r_2 = \lambda_1 \lambda_2$, so that

$$\frac{\partial r_2}{\partial \mu_1} = \frac{-r_2}{r_1} \frac{\partial r_1}{\partial \mu_1} \qquad (5.2.7)$$

Taking the partial derivative of $\overline{F}_{11}(t)$, we obtain by

the chain rule:

$$\frac{\partial \bar{F}_{11}(t)}{\partial \mu_1} = \frac{\partial \bar{F}_{11}(t)}{\partial r_1}\frac{\partial r_1}{\partial \mu_1} + \frac{\partial \bar{F}_{11}(t)}{\partial r_2}\frac{\partial r_2}{\partial \mu_1} \qquad (5.2.8)$$

Substituting from (5.2.5) and (5.2.7) yields:

$$\frac{\bar{F}_{11}(t)}{\partial \mu_1} = \frac{\partial r_1}{\partial \mu_1}\frac{e^{-r_1 t} r_2}{(r_1-r_2)^2}\,[2-2e^{(r_1-r_2)t}+(r_1-r_2)t(1+e^{(r_1-r_2)t})]$$

(5.2.9)

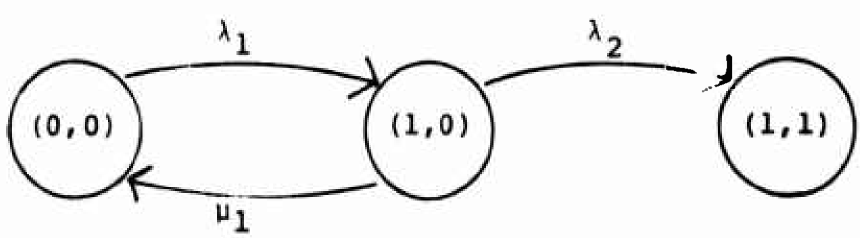
We note from (5.2.3) that $\frac{\partial r_1}{\partial \mu_1} > 0$. We conclude that all the terms outside the square brackets are strictly positive. We note that the term in the square brackets can be written as $[2-2e^X + X(1+e^X)]$ where $X = (r_1-r_2)t > 0$ if $t > 0$. Let $f(X) = 2-2e^X + X(1+e^X)$. We will show $f(X) > 0$, $X > 0$, and thus the theorem will be proven as desired. Note that $f(0) = 0$, $f'(0) = 0$, $f''(\underline{X}) > 0$, $X > 0$; and the desired result follows from Taylor's theorem with remainder.[42] End of proof of Theorem 5.2 A.

5.3 The "Lazy" Repairman Two Component Series System

In this section we improve the results of the last section. Specifically, we assume that the repairman is interrupted periodically and does no work on the system. These interruptions occur independently of the state of the system. Of course, failures in working components can occur during the interruptions. We show that, in this more general setting, if $\mu_1 < \mu_2$ the system is repaired stochastically more quickly by using policy π_1 while the repairman works than by using policy π_2 while the repairman works. If $\mu_1 = \mu_2$ the time under both policies is stochastically equal.

We then have the following model. The repairman works for the first time period of length t_1, is idle for the second time period of length t_2, works for the third time period of length t_3, is idle for the fourth time period of length t_4, and etc.

The evolution of the system state during odd periods and under policy π_1 is governed by a continuous time Markov chain with the following transition diagram,[24,37]



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and the evolution of the system state during even time

periods is governed by a continuous time Markov chain with the following transition diagram:

$$(0,0) \xleftarrow{\mu_1} (1,0) \qquad\qquad (1,1) \quad .$$

<u>Theorem 5.3 A</u>. Under the system evolution previously described, in which the repairman alternately works and is idle, the time to reach (1,1) from (0,0) under policy π_1 of section 5.2 is strictly stochastically less than the time to reach (1,1) from (0,0) under policy π_2 if $\mu_1 < \mu_2$. If $\mu_1 = \mu_2$, the times are stochastically identical.

<u>Proof</u>: We show that the probability that the system has been repaired at any time using policy π_1 is greater than the probability the system has been repaired at the same time using policy π_2. It is easily seen that we need only consider odd numbered time intervals, since repairs cannot occur during the even numbered intervals. We show that this is true for period 2n-1, regardless of length, by induction on n. Theorem 5.2 A shows that the claim is true for n = 1.

We now assume that the theorem is true for period 2n-1, and wish to show its truth for period 2n+1. At the end of period 2n-1, the probability vector for the system state under π_1 is

$$\begin{pmatrix} P_{\pi_1}(\underline{X}(\sum_{i=1}^{2n-1} t_i)=(0,0)) \\ \\ P_{\pi_1}(\underline{X}(\sum_{i=1}^{2n-1} t_i)=(1,0)) \end{pmatrix} = \begin{pmatrix} P_{11} \\ \\ P_{12} \end{pmatrix}, \qquad (5.3.1)$$

for some P_{11} and P_{12}.

Similarly under π_2, the probability vector for the system state is

$$\begin{pmatrix} P_{\pi_2}(\underline{X}(\sum_{i=1}^{2n-1} t_i)=(0,0)) \\ \\ P_{\pi_2}(\underline{X}(\sum_{i=1}^{2n-1} t_i)=(0,1)) \end{pmatrix} = \begin{pmatrix} P_{21} \\ \\ P_{22} \end{pmatrix}, \qquad (5.3.2)$$

for some P_{21} and P_{22}.

At the end of period 2n, the appropriate state vectors are

$$\begin{pmatrix} P_{\pi_1}(\underline{X}(\sum_{i=1}^{2n} t_i)=(0,0)) \\ \\ P_{\pi_1}(\underline{X}(\sum_{i=1}^{2n} t_i)=(1,0)) \end{pmatrix} = \begin{pmatrix} P_{11}+P_{12}(1-e^{-\mu_1 t_{2n}}) \\ \\ P_{12}e^{-\mu_1 t_{2n}} \end{pmatrix}, \qquad (5.3.3)$$

and

$$\begin{pmatrix} P_{\pi_2}(\underline{X}(\sum_{i=1}^{2n} t_i)=(0,0)) \\ \\ P_{\pi_2}(\underline{X}(\sum_{i=1}^{2n} t_i)=(0,1)) \end{pmatrix} = \begin{pmatrix} P_{21}+P_{22}(1-e^{-\mu_2 t_{2n}}) \\ \\ P_{22}e^{-\mu_2 t_{2n}} \end{pmatrix} . \quad (5.3.4)$$

Now introduce the following notation.

$\bar{F}_{11}(t)$ = P (repair time is greater than t from (0,0) under π_1 with repairman always working)

$\bar{F}_{12}(t)$ = P (repair time is greater than t from (1,0) under π_1 with repairman always working)

$\bar{F}_{21}(t)$ = P (repair time is greater than t from (0,0) under π_2 with repairman always working)

$\bar{F}_{22}(t)$ = P (repair time is greater than t from (0,1) under π_2 with repairman always working)

(5.3.5)

Let t be the elapsed time in period 2n+1. Letting $\bar{F}_1(t)$ be the probability the system is not yet repaired at this time under policy π_1, and $\bar{F}_2(t)$ be the probability the system is not yet repaired at this time under policy π_2, and using (5.3.3), (5.3.4), and (5.3.5) we obtain

$$\bar{F}_1(t) = [P_{11}+P_{12}(1-e^{-\mu_1 t_{2n}})]\bar{F}_{11}(t) + P_{12}e^{-\mu_1 t_{2n}}\bar{F}_{12}(t) \quad (5.3.6)$$

and

$$\bar{F}_2(t) = [P_{21}+P_{22}(1-e^{-\mu_2 t_{2n}})]\bar{F}_{21}(t) + P_{22}e^{-\mu_2 t_{2n}}\bar{F}_{22}(t) \quad . \tag{5.3.7}$$

Note that by induction hypothesis if $t_{2n} = 0$, then $\bar{F}_1(t) \leq \bar{F}_2(t)$ since period 2n+1 is then just a continuation of period 2n-1. Again equality occurs iff $\mu_1 = \mu_2$.

Now let $t_{2n} \to \infty$ in (5.3.6) and (5.3.7). We note by the induction hypothesis that $P_{11} + P_{12} \leq P_{21} + P_{22}$ since these represent the respective probabilities that repair occurs after the end of period 2n-1 under the two policies. Again equality occurs iff $\mu_1 = \mu_2$. The fact that $\bar{F}_{11}(t) \leq \bar{F}_{21}(t)$ was shown in section 5.2, so that as $t_{2n} \to \infty$, $\bar{F}_1(t) \leq \bar{F}_2(t)$.

Thus, for $t_{2n} = 0$ or $t_{2n} \to \infty$, $\bar{F}_1(t) \leq \bar{F}_2(t)$, with equality iff $\mu_1 = \mu_2$. We show that this implies there is no value of t_{2n} such that $\bar{F}_1(t) > \bar{F}_2(t)$.

Taking derivatives of (5.3.6) and (5.3.7) yields:

$$\frac{\partial \bar{F}_1(t)}{\partial t_{2n}} = \mu_1 P_{12}[\bar{F}_{11}(t)-\bar{F}_{12}(t)]e^{-\mu_1 t_{2n}} \tag{5.3.7}$$

and

$$\frac{\partial \bar{F}_2(t)}{\partial t_{2n}} = \mu_2 P_{22}[\bar{F}_{21}(t)-\bar{F}_{22}(t)]e^{-\mu_2 t} \quad . \tag{5.3.8}$$

We see that by (5.3.5) $\bar{F}_{11}(t) > \bar{F}_{12}(t)$ and

$\bar{F}_{21}(t) > \bar{F}_{22}(t)$ so that

$$\frac{\partial \bar{F}_1(t)}{\partial t_{2n}} > 0 \qquad \text{and} \qquad \frac{\partial \bar{F}_2(t)}{\partial t_{2n}} > 0 \quad .$$

Now assume there is a value of t_{2n}, such that when $t_{2n} = t'$, $\bar{F}_1(t) - \bar{F}_2(t) > 0$. We show this leads to a contradiction.

Consider the above expression as a function of t_{2n}.

$$G(t_{2n}) = \bar{F}_1(t) - \bar{F}_2(t) \tag{5.3.10}$$

Taking the derivative of (5.3.10) with respect to t_{2n} and using (5.3.8) and (5.3.9) we obtain

$$G'(t_{2n}) = k_1 e^{-\mu_1 t_{2n}} - k_2 e^{-\mu_2 t_{2n}} \tag{5.3.11}$$

where

$$k_1 = \mu_1 P_{12}[\bar{F}_{11}(t) - \bar{F}_{12}(t)] > 0$$

$$k_2 = \mu_2 P_{22}[\bar{F}_{21}(t) - \bar{F}_{22}(t)] > 0 \quad .$$

Since $G(0) \leq 0$ and $G(t') > 0$ we require for some $t_0 < t'$, $G'(t_0) > 0$. But (5.3.11) requires that $G'(t) > 0$, $t > t'$ since $\mu_1 \leq \mu_2$. We conclude that $G(t) > 0$, $t > t'$, a contradiction as $t \to \infty$.

If $\mu_1 < \mu_2$, by the same reasoning we cannot have $G(t') \geq 0$ since $G(0) < 0$.

Stochastic equality of the repair times follows when $\mu_1 = \mu_2$ since $\bar{F}_1(t) \leq \bar{F}_2(t)$ and $\bar{F}_2(t) \leq \bar{F}_1(t)$. End of inductive proof, and end of the proof of Theorem 5.3 A.

<u>Corollary</u>: When the periods t_i are random variables independent of the state of the system, Theorem 5.3 A still holds.

5.4 The n Component Series System

The results of the last section intuitively tell us that the optimal policy for an n component series system is to repair the components in order of increasing μ_i. Unfortunately, this general result seems difficult to prove. We can prove the following result.

Theorem 5.4 A. Suppose π is a candidate policy in the form of a list; i.e., suppose policy π always repairs the failed component closest to the top of some fixed list of the components. If the components on the list are not arranged in order of increasing failure rates, then π is not optimal.

Proof: Find two components on the list for π in which the failure rates are inverted. Call these components i and j, $\mu_i > \mu_j$.

Repair of the system from $\underline{0}$ under π will take the following form. First, in phase 1, repair all the components on the list above i and j. Then, in phase 2, repair components i and j, interrupting the repair of i and j to repair the set of components above i and j on the list if any should fail. Lastly, in phase 3, repair the rest of the components, interrupting if necessary.

However, by Theorem 5.3 A we know that the second phase (repairing i and j with interruptions) can be done

strictly faster if the order of repair of these two components is switched.

Thus a policy π' which uses π during phases 1 and 3 and switches the order of repair of components i and j during phase 2 will have an expected system repair time strictly less than the expected system repair time under policy π, contradicting the optimality of π. End of proof of Theorem 5.4 A.

Theorem 5.4 B. If an n component series system whose repairman is subject to the kind of interruptions described in section 5.3 always has a component, i, which may be optimally repaired whenever failed, then the optimal series policy is in the form of a list.

Proof: The proof is by induction on n. The theorem is clearly true for n = 2. For an n+1 component system, note that under a policy with component i always repaired, if possible, the repairman's availability to the n other components follows the interrupted repairman model of the last section. Thus, if the theorem is true for n components, it is also true for n+1 components. End of proof of Theorem 5.4 B.

Intuitively, the complete symmetry of the series system suggests that a list policy is optimal. However, we will see in section 6.3 that optimal policies for general

systems are not always expressible in the form of a list.

CHAPTER 6

Stochastically Identical Components

When all components are stochastically identical, i.e., $\lambda_i \equiv \lambda$, $\mu_i \equiv \mu$; certain symmetries in the structure function can often be exploited to show that certain actions are not optimal. Thus, by contradiction, the optimal action in a given state can often be obtained.

6.1 Permutation Operators

When all components are stochastically identical, the proper system evolution is maintained whenever working components are permuted, providing that the permutations depend only on the past history of the system. Similarly, the proper system evolution under a different policy is obtained if non-working components are permuted, provided that permutations depend only on the past history of the system.

We are particularly interested in permutations which take place at system state transitions.

Let the system evolution under a candidate policy π be on a probability space $(\Omega,\mathcal{F},P)$. Thus, given $\omega\epsilon\Omega$ we know $\underline{X}(t,\omega)$ for all $t \geq 0$. Without loss of generality, let the $n^{\underline{th}}$ transition for $\underline{X}(t,\omega)$ occur at $t_n(\omega)$ with $0 \overset{def}{=} t_o(\omega) < t_1(\omega) < t_2(\omega) < t_3(\omega) < \ldots$. This is possible since the measure of the set of points ω without this property is zero. Let the total number of transitions up to and including time t be $N(t,\omega)$.

A permutation operator, ρ, will permute the components of an n vector. Thus ρ can be specified by a permutation of the first n integers, $\rho = (n_1,\ldots n_n)$, $n_i \neq n_j$, $i \neq j$; and $n_i \epsilon (1,\ldots,n)$ with the understanding that $(\rho\underline{X})_i = \underline{X}_{n_i}$. Thus, if $\rho = (2,1,3)$, $\rho(X_1,X_2,X_3) = (X_2,X_1,X_3)$.

The identity permutation operator, I, will be defined to be $I = (1,2,\ldots,n)$.

For a permutation operator ρ, and an integer i,

$1 \leq i \leq n$ define $\rho(i) = C_1(\rho(1_i,\underline{0}))$.

We are particularly interested in a new system evolution $\underline{X}'$, whose transitions occur at the same times as the times of transition for $\underline{X}$, and whose state at a time of transition, $\underline{X}'(\omega,t_i(\omega))$, is obtained by permutation of $\underline{X}(\omega,t_i(\omega))$, the original state vector at the same time.

Given $\rho_i(\omega)$, $i = 0,1,\ldots$, permutation operators, define the permutation operator $\rho^n(\omega)$ as follows:

$$\rho^n(\omega) = \rho_n(\omega)\rho_{n-1}(\omega)\ldots\rho_0(\omega) \quad . \tag{6.1.1}$$

By convention $\rho^{-1}(\omega) = I$. Now define the permuted process $\underline{X}'(\omega,t)$ as follows:

$$\underline{X}'(\omega,t) = \rho^n(\omega)\underline{X}(\omega,t_n(\omega)), \quad t_n(\omega) \leq t < t_{n+1}(\omega) \quad . \tag{6.1.2}$$

<u>Theorem 6.1 A</u>. Let the system evolution under candidate policy π be on a space $(\Omega,\mathcal{F},P)$ with the properties previously described in this section. Let $\rho_i(\omega)$ depend only on $\underline{X}(\omega,t)$ for $t \leq t_i(\omega)$, with the property that $\rho_0(\omega)\underline{X}(\omega,0) = \underline{X}(\omega,0)$, and for $i = 0,\ldots$,

$$\rho_{i+1}(\omega)\rho^i(\omega)\underline{X}(\omega,t_{i+1}(\omega)) = \rho^i(\omega)\underline{X}(\omega,t_{i+1}(\omega)) \quad . \tag{6.1.3}$$

Then $\underline{X}'(\omega,t)$ given by (6.1.2) gives the proper system evolution on $(\Omega,\mathcal{F},P)$ for some non-stationary policy which depends on the past history of the system.

Proof: Let the component under repair under π' at (ω,t) when $t_i(\omega) \leq t < t_{i+1}(\omega)$ be constant and be the component which is constantly under repair under π in the same interval, permuted by $\rho^i(\omega)$. This is just $\rho^i(\omega)[\pi(\underline{X}(\omega,t_i(\omega)))]$.

The proper future system evolution under π', given the past history of the system, follows immediately from (6.1.2), and from the proper future system evolution of $\underline{X}$ given the past history of $\underline{X}$ guaranteed by the probability space $(\Omega,\mathcal{F},P)$.

More explicitly, a repair in the original process is accompanied by a repair in the primed process, and the failure of a particular component j in the original process is accompanied by the failure of the working component in the primed process associated with component j through permutation. The stochastic identity of the components insures that the proper law is followed for the primed process. End of proof of Theorem 6.1 A.

As an example of the use of Theorem 6.1 A, we show how the system evolution under all candidate policies can be mapped into the same probability space.

Let the system evolution under a candidate policy π be on the space $(\Omega,\mathcal{F},P)$ previously described. We show how to obtain $\rho_i(\omega)$ so that the system evolution under another candidate policy π' is given by (6.1.2).

For $\underline{X}(\omega,t_i(\omega)) = \underline{1}$, $\rho_i(\omega) = I$. For $\underline{X}(\omega,t_i(\omega)) \neq \underline{1}$, and $i = 0,1,...$, let $\rho_i(\omega)$ be the permutation operator which

exchanges the coordinates $\rho^{i-1}(\omega)[\pi(\underline{X}(\omega,t_i(\omega)))]$ and $\pi'[\rho^{i-1}(\omega)\underline{X}(\omega,t_i(\omega))]$. By construction, $\rho_i(\omega)$ exchanges coordinates which are zero in $\rho^{i-1}\underline{X}(\omega,t_i(\omega))$, and thus (6.1.3) is satisfied.

By the argument given in the proof of Theorem 6.1 A, the component under repair under π' for (ω,t), $t_i(\omega) \leq t < t_{i+1}(\omega)$, is $\rho_i(\omega)\rho^{i-1}(\omega)(\pi(\underline{X}(\omega,t)))$. By the construction of $\rho_i(\omega)$, this is just $\pi'[\rho^{i-1}(\omega)\underline{X}(\omega,t_i(\omega))]$. However, by (6.1.2) and (6.1.3), this is just $\pi'(\underline{X}'(\omega,t))$.

Thus, by Theorem 6.1 A, $\underline{X}'(\omega,t)$ given by (6.1.2) is the proper system evolution under candidate policy π'.

Notice that the above argument is correct even if $\rho_i(\omega)$ permutes coordinates which are equal to 1 in $\rho^{i-1}(\omega)\underline{X}(\omega,t_i(\omega))$.

6.2 Contradiction of Optimal Policies

The primary use of Theorem 6.1 A is to contradict the optimality of candidate policies.

Assume candidate policy π is optimal. We attempt to choose the $\rho_i(\omega)$ in such a manner that the evolution under (6.1.2) of $\underline{X}'$ strictly dominates the evolution under the original policy of $\underline{X}$. That is, we choose $\rho_i(\omega)$ in Theorem 6.1 A so that if $\phi(\underline{X}(\omega,t)) = 1$, then $\phi(\underline{X}'(\omega,t)) = 1$ and that on a set of non-zero measure in ω and t, $\phi(\underline{X}'(\omega,t)) = 1$ and $\phi(\underline{X}(\omega,t)) = 0$. This, of course, implies the existence of a policy π' which is strictly better than π, and thus contradicts the optimality of π.

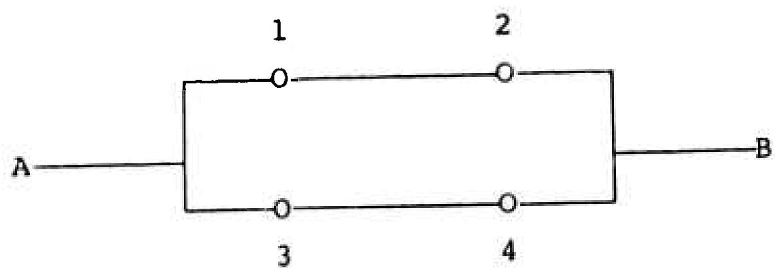
6.3 An Example

In the following example we show the use of the previously described method for determination of the optimal policy.

Consider a four component system whose structure function is given by:

$$\phi((X_1,X_2,X_3,X_4)) = 1-(1-X_1X_2)(1-X_3X_4) \quad . \tag{6.3.1}$$

Such a structure function may be represented pictorially as a network as follows:



with the interpretation that the system works iff there is a path from A to B through working components.

Intuitively, due to the symmetry present, the optimal policy is of the following form: repair any failed component when none or two of the components work. When one component works, repair the component whose repair causes the system to work. For other states there is no choice allowed in a candidate policy.

To prove that such a policy is optimal, we contradict the optimality of any candidate policy not of the given

form. All candidate policies of the given form are then easily seen to have stochastically equivalent values of ϕ.

We now show that the optimal policy cannot repair component 3 in the state (1,0,0,0).

Let π be a candidate policy which repairs component 3 in the state (1,0,0,0). Let the evolution under π be described on a probability space $(\Omega,\mathcal{F},P)$. Let π' be a candidate policy of the type which was described to be intuitively optimal.

Use the method described at the end of section 6.1 to map the system evolution under policy π' into $(\Omega,\mathcal{F},P)$. Remember that $\rho_i(\omega)$ may arbitrarily permute working coordinates of $\rho^{i-1}(\omega)\underline{X}(\omega,t_i(\omega))$. We now give a rule for permuting such coordinates.

If $X_1(\omega,t_i(\omega)) = X_2(\omega,t_i(\omega)) = 1$, choose $\rho_i(\omega)$ such that either

$$\rho_i(\omega)\rho^{i-1}(\omega)(1,1,0,0) = (1,1,0,0)$$

or

$$\rho_i(\omega)\rho^{i-1}(\omega)(1,1,0,0) = (0,0,1,1) \quad .$$

If $X_3(\omega,t_i(\omega)) = X_4(\omega,t_i(\omega)) = 1$, choose $\rho_i(\omega)$ such that either

$$\rho_i(\omega)\rho^{i-1}(\omega)(0,0,1,1) = (1,1,0,0)$$

or

$$\rho_i(\omega)\rho^{i-1}(\omega)(0,0,1,1) = (0,0,1,1) \quad .$$

We have yet to show that it is always possible to so choose $\rho_i(\omega)$.

Theorem 6.3 A. It is always possible to choose $\rho_i(\omega)$ as described immediately previously, and $\phi(\underline{X}'(\omega,t_i(\omega)) \geq \phi(\underline{X}(\omega,t_i(\omega))$ for every i.

Proof: The proof is by induction on i. For i = 0, $\underline{X}'(\omega,0) = \underline{X}(\omega,0)$ and $\rho_0(\omega)$ need not permute working coordinates of $\underline{X}(\omega,0)$ so that the theorem holds in this case.

Now assume the theorem holds for i. We show that the theorem is true for i+1.

The transition at $t_{i+1}(\omega)$ can either be a repair or a failure. If the transition at $t_{i+1}(\omega)$ is a repair and $\phi(\underline{X}(\omega,t_{i+1}(\omega)) = 1$, then $|C_1(\underline{X}(\omega,t_{i+1}(\omega)))| \geq 2$ and thus $|C_1(\underline{X}'(\omega,t_{i+1}(\omega)))| \geq 2$ and by the nature of π', $\phi(\underline{X}'(\omega,t_{i+1}(\omega)))= 1$. If the transition at $t_{i+1}(\omega)$ is a failure and $\phi(\underline{X}(\omega,t_{i+1}(\omega))) = 1$ then by construction of ρ_i, either $X_1'(\omega,t_{i+1}(\omega)) = X_2'(\omega,t_{i+1}(\omega)) = 1$, or $X_3'(\omega,t_{i+1}(\omega)) = X_4'(\omega,t_{i+1}(\omega)) = 1$; so that $\phi(\underline{X}'(\omega,t_{i+1}(\omega)) = 1$.

Thus, regardless of the type of the $i+1^{\underline{st}}$ transition, $\phi(\underline{X}'(\omega,t_{i+1}(\omega)) = 1$ if $\phi(\underline{X}(\omega,t_{i+1}(\omega)) = 1$. This condition assures domination of the primed process over the unprimed process at $(\omega,t_{i+1}(\omega))$ and also ensures that $\rho_{i+1}(\omega)$ can be chosen as desired. End of induction.

Strict domination follows when repairs occur for $\underline{X} = (1,0,0,0)$. End of the proof of theorem 6.3 A.

Corollary: A candidate policy which does not repair component 2 in state (1,0,0,0) is not optimal.

This follows from the last theorem and the symmetry of components 3 and 4.

Corollary: The intuitive optimal policy described earlier is optimal.

This follows from the last corollary, the symmetry of components, and the easily proven fact that all candidate policies of the intuitively optimal form can be mapped into the same probability space with pointwise equivalent values of ϕ. Such a proof is similar to the proof of Theorem 6.3 A, and is omitted.

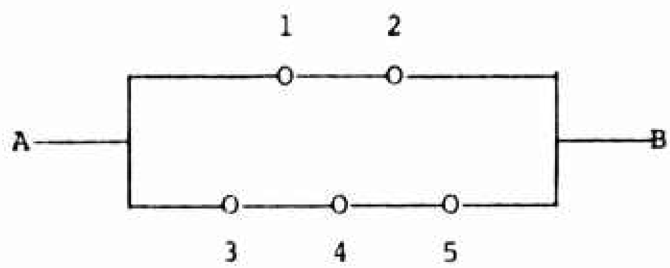
We note that the optimal policy for the system described in this section contradicts the hypothesis that all optimal policies can be expressed in the form of a list (see section 5.4).

When components 2 and 4 are failed in the state (1,0,0,0) it is strictly optimal to repair component 2. When components 2 and 4 are failed in the state (0,0,1,0) it is strictly optimal to repair component 4.

6.4 A Second Example

The previous kind of argument does not always find the optimal actions for all states of an arbitrary structure function. Often, a substantial number of candidate policies can be eliminated and the solution for systems can be considerably simplified.

For example, consider the following 5 component system



with the following structure function

$$\phi((X_1,X_2,X_3,X_4,X_5)) = 1-(1-X_1X_2)(1-X_3X_4X_5) \quad . \qquad (6.4.1)$$

By arguments based on the symmetry similar to those given in the last section, the state of the system can be summarized by the following two coordinates, $(X_1+X_2,\ X_3+X_4+X_5)$. In the same manner, there are only two possible actions to consider in any state which are 1) repair one of the first two components, and 2) repair one of the last three components. We can also prove by arguments similar to those of the last section that the following actions are optimal.

State	Action
(0,0)	1
(1,0)	1
(2,0)	2
(0,1)	1
(1,1)	1
(2,1)	2
(0,2)	?
(1,2)	1
(2,2)	2
(0,3)	1
(1,3)	1
(2,3)	N.A.

Thus, the only state in which the optimal action is not readily obtainable is (0,2). This is because action 2 dominates action 1 for small numbers of transitions whereas action 1 dominates action 2 for some larger numbers of transitions.

Nevertheless, we need only compare 2 candidate policies. After much algebra, we find that action 2 is optimal in state (0,2) regardless of λ and μ. (This is the highly reliable and highly unreliable optimal action.)

There are $\prod_{j=2}^{n} j^{\binom{n}{j}}$ candidate policies to consider for an n component system. When n = 5 this number is $5 \cdot 4^5 \cdot 3^{10} \cdot 2^5 \approx 10^{10}$, and thus we have provided considerable simplification.

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